LECTURE 3

DRE 7007

MATHEMATICS

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PLAN:

1. Functions and continuity.
2. Derivatives and partial derivatives.

Reading:

[FMEA] 13.3

[ME] 13.4, 14

[S] 1.4

1 Functions and continuity

A <u>function</u> $f: X \to Y$ is a rule that assigns a unique value $y = f(x) \in Y$ to every element $x \in X$. The set ~~X~~ is called the <u>domain</u> and Y is called the <u>codomain</u> of f.

Most functions we consider are functions $f: D \to \mathbb{R}$, where the domain $D \subseteq \mathbb{R}^n$ and the codomain is $\mathbb{R}$.

<u>Example</u>: $f(x,y) = e^{xy} - 1$, which can be written $\begin{aligned} f: \mathbb{R}^2 &\to \mathbb{R} \\ (x,y) &\mapsto e^{xy} - 1 \end{aligned}$

$g(x,y) = \frac{1}{x^2+y^2}$, $(x,y) \neq (0,0)$ which can be written $\begin{aligned} g: D &\to \mathbb{R} \\ (x,y) &\mapsto \frac{1}{x^2+y^2} \end{aligned}$ with $D = \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}$

But also this is a function:

$$\begin{aligned} C([0,1], \mathbb{R}) &\longrightarrow \mathbb{R} \\ f &\longmapsto \int_0^1 f(x)\,dx \end{aligned}$$

Such functions, where the domain consists of functions, are often called <u>operators</u>.

Let $f: X \to Y$ be a function, where X and Y are metric spaces. We say that f is continuous at $x \in X$ if the following condition holds:

For any $\varepsilon > 0$, there is a $\delta > 0$ such that
$$x' \in X \text{ with } d(x', x) < \delta \Rightarrow d(f(x'), f(x)) < \varepsilon$$

In other words, $x' \in B(x, \delta) \Rightarrow f(x') \in B(f(x), \varepsilon)$

We say that <u>f is continuous</u> if it is continuous at x for all $x \in X$.

Consequences of continuity:

Theorem:

If $f: X \to Y$ is continuous and $K \subseteq X$ is compact, then

$$f(K) = \{ y = f(x) : x \in K \} \subseteq Y$$

is also compact.

Theorem: (Weierstrass) (Extreme value thm.)

If $f: D \to \mathbb{R}$ is continuous, where $D \subset \mathbb{R}^n$ is a closed and bounded set, then f has a <u>maximum</u> and a <u>minimum</u>.

Proof:

D is compact, so $f(D)$ is compact. But the compact sets in $\mathbb{R}$ are closed and bounded. Let $M = \sup f(D)$ and $m = \inf f(D)$. Since $f(D)$ is closed and bounded, there are x_{max} and x_{min} in D such that $f(x_{max}) = M$, $f(x_{min}) = m$.

How to determine if a function is continuous

Facts:
* All "elementary" functions are continuous (polynomials, rationals, exponentials, logarithms).
* Sums, products and quotients of continuous functions are cont.
* Compositions of cont. functions are cont.

Ex:

$$f(x) = \begin{cases} x+1, & x \geq 0 \\ e^x, & x < 0 \end{cases}$$

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

Function spaces

Let $D \subseteq X$ be a subset of a metric space X. Define

$$C(D) = \{ f : D \to \mathbb{R} \mid f \text{ is continuous} \}$$

Facts:

i) $C(D)$ is a vector space

ii) The sup norm is a norm on $C(D)$, given by

$$\|f\|_{\sup} = \sup_{x \in D} |f(x)| = \sup \{ |f(x)| : x \in D \}$$

for $f \in C(D)$.

iii) If D is compact, then $C(D)$ with sup norm is a complete metric space.

Note: Alternative defn. of continuity

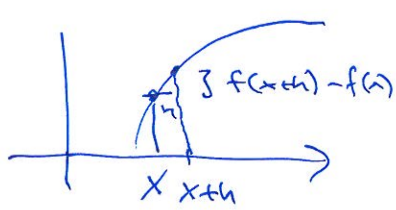
$f : X \to Y$ cont. at $x \in X$ if and only if the following condition holds:

$\boxed{\text{For any sequence } (x_i) \text{ in } X \text{ with } \lim (x_i) = x, \text{ we have } \lim (f(x_i)) = f(x).}$

② Derivatives and partial derivatives

Recall: If $f: \mathbb{R} \to \mathbb{R}$ is a function in one variable and $x \in \mathbb{R}$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



Let $f: D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$ is an open set. For any $\underline{x} \in D$, the partial derivatives

$$f_i'(\underline{x}) = \frac{\partial f}{\partial x_i}(\underline{x}) = \lim_{h \to 0} \frac{f(\underline{x} + \underline{e_i} \cdot h) - f(\underline{x})}{h}, \quad \text{where } \underline{e_i} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

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if this limit exists.

For any $x \in D$, the total derivative is $Df(\underline{x})$ if there is a $1 \times n$-matrix $Df(\underline{x}) = A$ such that the following condition holds:

For any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$y \in D, \ \|y - x\| < \delta \implies \|f(y) - f(x) - A \cdot (y - x)\| < \varepsilon \cdot \|y - x\|$$

In other words,

$$\lim_{y \to x} \frac{\|f(y) - f(x) - A(y-x)\|}{\|y - x\|} = 0 \qquad A = \left(\frac{\partial f}{\partial x_1}(x) \ \frac{\partial f}{\partial x_2}(x) \ \cdots \ \frac{\partial f}{\partial x_n}(x) \right)$$

If $Df(\underline{x})$ exists, we say that f is differentiable in x. If it is differentiable for all $x \in D$, f is called differentiable.

Facts:

i) If f is differentiable at $x \in D$, then all partial derivatives $\frac{\partial f}{\partial x_i}(x)$ exists, and $Df(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right)$.

ii) If all partial derivatives $\frac{\partial f}{\partial x_i}(x)$ exists and are continuous at x, then f is differentiable and $Df(\underline{x}) = \left(\frac{\partial f}{\partial x_1}(x) \ \cdots \ \frac{\partial f}{\partial x_n}(x) \right)$.

f is called a C^1 function if $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ exists and are continuous

Facts:

i) Sums of differentiable functions are differentiable

ii) Composition of differentiable functions are differentiable.

Assume that $f: D \to \mathbb{R}$ is a C^1 function. We denote the j'th partial derivative of $(\partial f / \partial x_i): D \to \mathbb{R}$ by $\frac{\partial^2 f}{\partial x_i \partial x_j} = f''_{ij}$ if it exists.

We say that f is twice differentiable at x, with second derivative

$$D^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

in that case. The matrix $D^2 f(x)$ is also called the Hessian of f.

We say that f is C^2 if $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exists and is continuous for all i,j.

Theorem:

If $f: D \to \mathbb{R}$ is C^2, then the Hessian $D^2 f(x)$ is a symmetric matrix.