LECTURE 4

DRE 7007

EIVIND ERIKSEN 28/08 2012

MATHEMATICS

PLAN:

1. Convex sets
~~2. Convex functions~~
2. Separation results
3. Convex/concave functions

Reading:

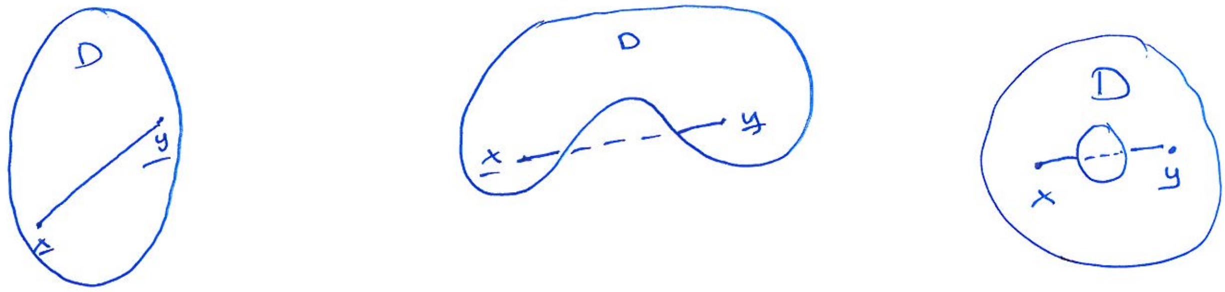
[FMEA] 2.2-2.3, 13.5-13.6

[ME] 21.1 -21.2

[S] 1.2, 1.6, 7

① Convex sets

A subset $D \subseteq \mathbb{R}^n$ is convex if for any $\underline{x}, \underline{y} \in D$, the straight line segment $[\underline{x}, \underline{y}] = \{\underline{z} \in \mathbb{R}^n : \underline{z} = \lambda \cdot \underline{x} + (1-\lambda) \cdot \underline{y},\ \lambda \in [0,1]\} \subseteq D$.



convex set   not convex set   not convex

Facts:

i) If S, S' are convex sets in $\mathbb{R}^n$, then

$$S + S' = \{s + s' : s \in S, s' \in S'\} \subseteq \mathbb{R}^n$$

is convex.

ii) If $(S_i)_i$ is a collection of convex sets, then $\bigcap_i S_i$ is convex.

② Separation theorems

Let $\underline{p} \neq \underline{0}$ be a vector in $\mathbb{R}^n$. Then the set

$$H = \{\underline{x} \in \mathbb{R}^n : \underline{p} \cdot \underline{x} = a\} \subseteq \mathbb{R}^n$$

is called a hyperplane, and we write $H = H(\underline{p}, a)$.

Two sets $D, \mathcal{E} \subseteq \mathbb{R}^n$ are separated by the hyperplane $H(\underline{p}; a) = H$ if $D, \mathcal{E}$ lie on opposite sides of H; i.e.

$$\underline{p} \cdot \underline{y} \geq a \quad \text{for all } y \in D$$
$$\underline{p} \cdot \underline{y} \leq a \quad \text{— " —} \ y \in \mathcal{E}$$

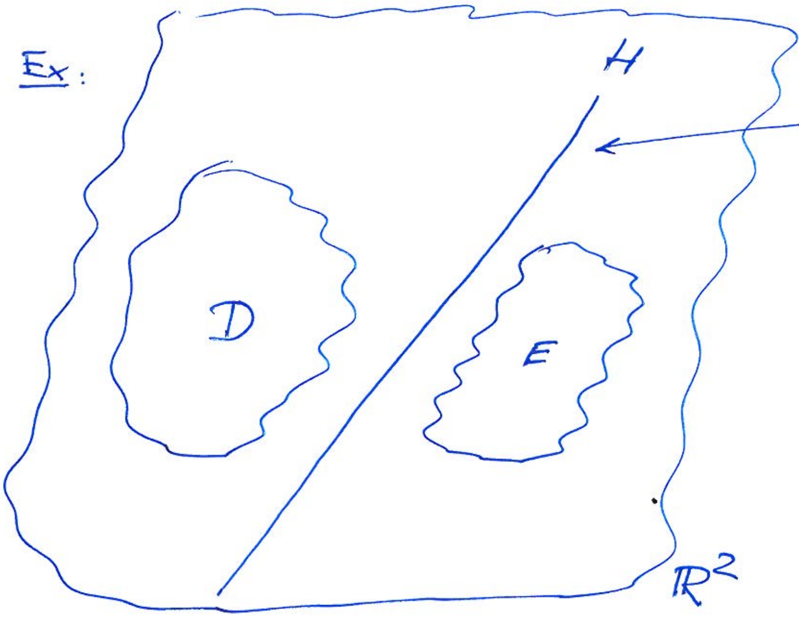
(or the other way around).

Notice that a hyperplane has equation

$$\underline{p} \cdot \underline{x} = a \iff p_1 x_1 + p_2 x_2 + \dots + p_n x_n = a$$

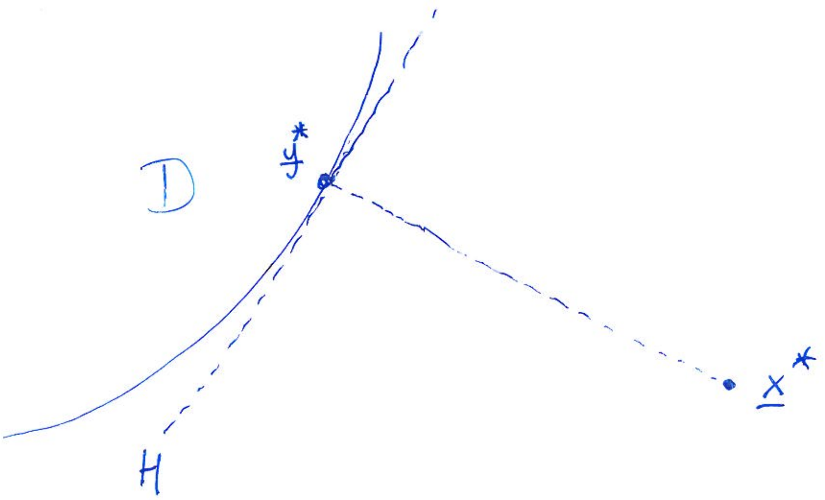
This a linear equation. If $n=2$, a hyperplane is a <u>line</u> and if $n=3$, a hyperplane is a <u>plane</u>.

<u>Ex</u>:



hyperplane that Separates D, E.

<u>Thm</u>:
If $D \subseteq \mathbb{R}^n$ is non-empty and convex set, and $\underline{x}^* \in \mathbb{R}^n \setminus D$, then there is a hyperplane $H = H(\underline{p}, a)$ that separates D and $\{\underline{x}^*\}$, with $\underline{p} \neq \underline{0}$. It is possible to choose $\|\underline{p}\| = 1$.



$y^* \in D$ s.t. $d(\underline{x}^*, y^*)$ is minimal

$H: \underline{p} \cdot \underline{x} = a$ with

$\underline{p} = y^* - \underline{x}^*$

$a = \underline{p} \cdot y^*$

Thm:

If $D, E \subseteq \mathbb{R}^n$ are convex s.t. $D \cap E = \emptyset$, then there is a hyperplane $H = H(\underline{p}, a)$ with $\underline{p} \neq \underline{0}$ that separates D, E. We may choose $\|p\| = 1$.

Proof:

Let $F = D + (-E)$, which is convex. Since $D \cap E = \emptyset$, $0 \notin F$.

Since $\underline{p} \cdot \underline{z} \geq \underline{p} \cdot \underline{0} = 0$ for all $\underline{z} = \underline{x} - \underline{y} \in F$, it follows that we have $\underline{p} \cdot x \geq \underline{p} \cdot y$ for all $\underline{x} \in D, \underline{y} \in E$. □

3 Convex / concave functions

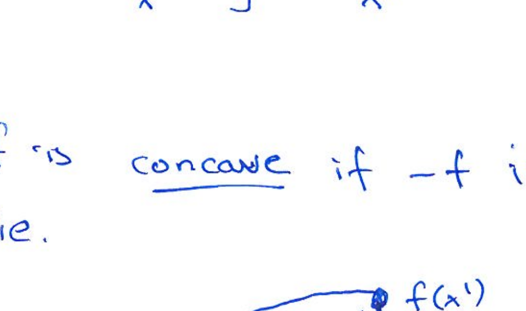
Let $f: D \to \mathbb{R}$ be a function defined on a convex set $D \subseteq \mathbb{R}^n$.

Then f is convex if the following condition holds:

For any $x, x' \in D$ and any $y = \lambda \cdot x + (1-\lambda) \cdot x' \in [x, x']$ (with $\lambda \in [0,1]$), we have

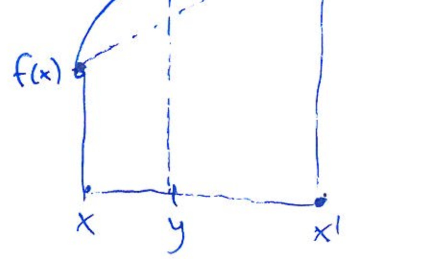
$$f(y) \leq \lambda \cdot f(x) + (1-\lambda) \cdot f(x') \in [f(x), f(x')]$$

"the straight line segment from $(x, f(x))$ to $(x', f(x'))$ lies over (or on) the graph of f"



Strict concave/convex if $<$ resp. $>$

f is concave if $-f$ is convex, ie.



" the straight line segment from $(x, f(x))$ to $(x', f(x'))$ lies under (or on) the graph of f "

__Thm:__

Let f be a C^2 function on a subset $D \subseteq \mathbb{R}^n$ that is open and convex. Then we have

$$f \text{ concave} \iff D^2 f(\underline{x}) \text{ negative semidefinite for all } \underline{x} \in D$$
$$f \text{ convex} \iff D^2 f(\underline{x}) \text{ positive ——— '' ———}$$

Similarly strict concave/convex correspond to pos./neg. definite.

__Facts:__

If $f: D \to \mathbb{R}$ is convex or concave on an open convex set $D \subseteq \mathbb{R}^n$, then f is continuous. Moreover, is differentiable for almost all $x \in D$, and C^1 in the points where it is differentiable.

Quasi-convex and quasi-concave functions

Let $f: D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$ is a convex set.

We say that f is

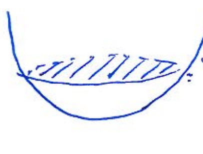
__quasi-concave__ if $U_f(a) = \{x \in D : f(x) \geq a\}$ is a convex set for all $a \in \mathbb{R}$

__quasi-convex__ if $L_f(a) = \{x \in D : f(x) \leq a\}$ is a convex set for all $a \in \mathbb{R}$

$$\boxed{\begin{array}{l} \text{convex} \Rightarrow \text{quasi-convex} \\ \text{concave} \Rightarrow \text{quasi-concave} \end{array}}$$

__Ex:__ $f(x,y) = x^2 + y^2$ defined on $\mathbb{R}^2$ is a typical convex fn.

$U_f(a) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq a\}$ is convex.



$\leftarrow x^2 + y^2 \leq a$