

LECTURE 5

EIVIND ERIKSEN

AUG 30 2012

DRE 7007

MATHEMATICS

PLAN:

- ① OPTIMIZATION
 - statement of problem - definitions
- ② UNCONSTRAINED OPTIMIZATION
 - first order conditions
 - second order conditions
 - concave / quasi-concave fu's.

Reading:

[FMEA] 3.1-3.2

[ME] 17,

[S] 2, 4, 7.3-7.6, 8.4-8.7

NEW PLACE AND TIME:

Thu 06/09	14.00-16.00	C2-095
Fri 07/09	14.00-16.00	C2-095

Deadline Assignments:

Mon 17/09 at 12.00

① Optimization

Let $f: D \rightarrow \mathbb{R}$ be a function, with $D \subseteq \mathbb{R}^n$. We consider the problems

$$\max_{x \in D} f(x) = \max \{f: D \rightarrow \mathbb{R}\} = \max f(x) \text{ sub. to } x \in D$$

$$\min_{x \in D} f(x) = \min \{f: D \rightarrow \mathbb{R}\} = \min f(x) \text{ sub. to } x \in D$$

A solution to the max problem is an $x^* \in D$ s.t. $f(x^*) \geq f(x)$ for all $x \in D$. Then x^* is also called a maximizer. Similar for min problems.

Fact:

$\min_{x \in D} f(x)$ and $\max_{x \in D} -f(x)$ have the same solutions

We write $\arg \max \{f(x) : x \in D\}$ for the set of maximizers. This set can be empty or have several points.

The attainable values of $\max_{x \in D} f(x)$ is the set $f(D) \subseteq \mathbb{R}$.

The max problem has a solution if and only if $\sup f(D) \in f(D)$.

D : constraint set
 f : objective function

Fact: If $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing function (i.e. such that $x < y \Rightarrow \alpha(x) < \alpha(y)$), then the max problem

$$\max_{x \in D} f(x)$$

and

$$\max_{x \in D} \alpha(f(x))$$

have the same solution.

Ex: $\max x^2 + y^2$ subj. to $2x^2 + 3y^2 \leq 6$

and

$\max \sqrt{x^2 + y^2}$ subj. to $2x^2 + 3y^2 \leq 6$

Same solution since

$x \mapsto \sqrt{x}$

is strictly increasing.

First problem is simpler.

Existence of solutions:

f cont., D compact

[W.]

\Rightarrow at least one solution of

$\max_{x \in D} f(x)$

f not cont. or D not compact

\Rightarrow the problem may or may not have solutions

② Unconstrained optimization

Let $\max_{x \in D} f(x)$ be a max problem. If x^* is a maximizer, it is called unconstrained if $x^* \in D$ is an interior point of D .

$x^* \in D$ interior point $\iff x^* \notin \partial D$
($x^* \in D - \partial D$)

(i.e. not a boundary point)

Case 1: $D = \mathbb{R}^n$ or $D \subseteq \mathbb{R}^n$ open \Rightarrow all maximizers are unconstrained

Case 2: $D \subseteq \mathbb{R}^n$ not open \Rightarrow there may be
- unconstrained maximizers (interior)
- constrained maximizers (boundary)

Defn: A point $x \in D$ is a local maximum of f on D if $f(x) \geq f(y)$ for all $y \in D \cap B(x, r)$ for some small $r > 0$ (ie for all $y \in D$ close to x). Similar for local minimum.

Note:

maximum \Rightarrow local maximum

Defn: $x \in D$ is called stationary point if $Df(x) = 0$, i.e. if $\frac{\partial f}{\partial x_1} = \dots = \frac{\partial f}{\partial x_n} = 0$, and a critical point if x is either stationary or $Df(x)$ does not exist (f not differentiable at x).

First order conditions:

x^* unconstrained local max (or min)
 $\Rightarrow x^*$ is a critical point

The important thing is if $Df(x)$ exists, not that f is C^1 at x .

Either $Df(x^*) = 0 \Leftrightarrow \frac{\partial f}{\partial x_1}(x^*) = \dots = \frac{\partial f}{\partial x_n}(x^*) = 0$
 or $Df(x^*)$ does not exist

Second order conditions:

If f is C^2 on D and $x \in D \cap \partial D$ interior pt. of D then

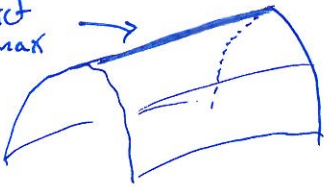
i) x local Max (min) $\Rightarrow D^2f(x)$ negative semidefinite (positive)

ii) $Df(x) = 0$ and $D^2f(x)$ negative definite $\Rightarrow x$ strict local max (min)

Implication, not equivalence in i)

local max but not strict
 local max looks like this

nonstrict local max



corresponds to $D^2f(x)$ negative semidefinite but not negative definite

Convex/concave:

If $f: D \rightarrow \mathbb{R}$ is concave, then

- i) local max \Rightarrow global max
- ii) $\arg \max \{f(x) : x \in D\}$ is empty or convex

If f is strictly concave, then

$\arg \max \{f(x) : x \in D\}$ is empty or a single point

Quasiconvex/concave:

If $f: D \rightarrow \mathbb{R}$ is strictly quasiconcave, then

- i) local max \Rightarrow global max
- ii) $\arg \max \{f(x) : x \in D\}$ is empty or a point