LECTURE 6

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MATHEMATICS

PLAN:

① CONSTRAINED OPTIMIZATION
- LAGRANGE - KUHN TUCKER

Reading:

[FMEA] 3.3-3.6
[ME] 18-19
[S] 5-6, 7.7, 8.8

Let $f: D \to \mathbb{R}$ be a function, defined on $D \subseteq \mathbb{R}^n$.
Consider

$$\max_{x \in D} f(x)$$

A point $x^* \in D$ is a constrained maximum if $x^* \in D$ is a maximizer such that $x^* \in \partial D$.

Examples:

i) min xyz when $\begin{cases} x - y + 2z = 3 \\ x + y = 3 \end{cases}$

$f(x,y,z) = xyz$, $D = \left\{ (x,y,z) : \begin{matrix} x - y + 2z = 3 \\ x + y = 3 \end{matrix} \right\}$

In this case, $\partial D = D$.

ii) min xyz when $\begin{cases} x - y + 2z \geq 3 \\ x + y \geq 3 \end{cases}$

$$D = \underbrace{(D \cap \partial D)}_{\text{boundary}} \cup \underbrace{(D \setminus \partial D)}_{\text{interior}}$$

Case I: Equality constraints.

Standard form

$$\max_{\min} f(x_1, \dots, x_n) \text{ subj. to. } \begin{cases} g_1(\underline{x}) = b_1 \\ g_2(\underline{x}) = b_2 \\ \vdots \\ g_k(\underline{x}) = b_k \end{cases}$$

$(k < n)$

idea: dim D is $n-k$

Non-degenerate constraint qualification (NDCQ)

$$\operatorname{rk} \begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \cdots & \partial g_1/\partial x_n \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 & \cdots & \\ \vdots & & & \\ \partial g_k/\partial x_1 & \partial g_k/\partial x_2 & \cdots & \end{pmatrix} = k$$

Are there points $x \in D$ such that NDCQ is not satisfied at x, i.e.,

$$\operatorname{rk} \left(\partial g_i/\partial x_j (x) \right) < k \quad ?$$

Lagrange method:

Consider max/min $f(\underline{x})$ subj. to $\begin{cases} g_1(\underline{x}) = b_1 \\ \vdots \\ g_k(\underline{x}) = b_k \end{cases}$

Thm:

If $\underline{x}^*$ is a solution to the max/min problem and if NDCQ is satisfied at $\underline{x}^*$, then the following condition holds:

(LC) If $L(\underline{x}, \underline{\lambda}) = f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \lambda_2 g_2(\underline{x}) - \dots - \lambda_k g_k(\underline{x})$,

then for some $\underline{\lambda}^* = \lambda_1^*, \dots, \lambda_k^*$.

FOC →
$$\boxed{\begin{aligned} \frac{\partial L}{\partial x_1}(x^*; \lambda^*) &= 0 \\ \frac{\partial L}{\partial x_2}(x^*; \lambda^*) &= 0 \\ &\vdots \\ \frac{\partial L}{\partial x_n}(x^*; \lambda^*) &= 0 \end{aligned}}$$

(constraints)

Cookbook:

i) Find all points $(\underline{x}^*, \underline{\lambda}^*)$ that satisfies FOC + C.

ii) Find all points $\underline{x}^*$ that does not satisfy NDCQ.

If there is a global max/min, then it must be one of the points in i) or ii)

To find out if there is a solution (global max & min), try a) Weierstr. thm b) AD HOC.

Sufficient conditions using convexity:

Assume that $(\underline{x}^*, \lambda^*)$ that satisfy FOC + C.

Consider the function

$$L(\underline{x}, \lambda^*) \quad \text{as a function in } (x_1, \dots, x_n) = \underline{x}$$

Then:

$$D^2 L(\underline{x}, \lambda^*) \text{ concave} \Rightarrow \underline{x}^* \text{ is max.}$$

$$D^2 L(\underline{x}, \lambda^*) \text{ convex} \Rightarrow \underline{x}^* \text{ is min.}$$

Interpretation of λ:

Given $\underline{b} = (b_1, b_2, \dots, b_k)$, let $\underline{x}^*(b)$ be a maximizer (minimizer) for the given $\underline{b}$ and let $f^*(\underline{b}) = f(x^*(\underline{b}))$ be the value at the max (min) for the given $\underline{b}$.

If $\underline{x}^*(b)$ satisfy FOC when $\underline{\lambda} = \underline{\lambda}^*$, then

$$\lambda_i^* = \frac{\partial f^*(\underline{b})}{\partial b_i}$$

"λ_i^* is the rate of change of the value function $f^*(\underline{b})$ when b_i is changed"

i.e. $\lambda_i > 0$ means that $f(\underline{b})$ increases when b_i increases

$\lambda_i < 0$ — " — decreases when b_i — " —

2) Case: Inequality constraints

$$\max f(\underline{x}) \text{ subj to. } \begin{cases} g_1(x) \le b_1 \\ g_2(x) \le b_2 \\ \vdots \\ g_k(\underline{x}) \le b_k \end{cases}$$

Standard form used in Kuhn-Tucker formulation..

A constraint $g_i(\underline{x}) \le b_i$ is
binding if $g_i(\underline{x}) = b_i$
non-binding if $g_i(\underline{x}) < b_i$

Thm:

If $\underline{x}^*$ is a solution to a standard K-T problem, and if $\underline{x}^*$ satisfy the NDCQ, then the following conditions hold:

If $L(\underline{x}, \underline{\lambda}) = f(\underline{x}) - \lambda_1 \cdot g_1(x) - \lambda_2 g_2(\underline{x}) \dots - \lambda_k g_k(x)$

then we have:

FOC
$$\boxed{\begin{aligned} \frac{\partial L}{\partial x_1}(x^*, \lambda^*) &= 0 \\ &\vdots \\ \frac{\partial L}{\partial x_n}(x^*, \lambda^*) &= 0 \end{aligned}}$$
for some $\underline{\lambda}^*$

and

CSC
$$\boxed{\begin{aligned} &\lambda_1^* \ge 0 \quad \text{and} \quad \lambda_1^* \cdot (g_1^*(\underline{x}^*) - b_1) = 0 \\ &\lambda_2^* \ge 0 \quad \text{and} \quad \lambda_2^* \cdot (g_2(\underline{x}^*) - b_2) = 0 \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\lambda_k^* \ge 0 \quad \text{and} \quad \lambda_k^* (g_k(x^* - b_k)) = 0 \end{aligned}}$$
$\Leftrightarrow$

For $i = 1, 2, \dots, k$, we have

$\lambda_i \ge 0$ and

$\lambda_i = 0$ if $g_i(\underline{x}) < b_i$ is not bind.

NDCQ in Kuhn-Tucker case:

Let $\underline{x} \in D = \{\underline{x} : g_1(\underline{x}) \le b_1, \ldots, g_k(\underline{x}) \le b_k\}$, and set

$$B = \{i \in \{1, 2, \ldots, k\} : g_i(\underline{x}) = b_i\}$$

= the set of indices corresponding to binding constraints at the given point $\underline{x}$.

Then the NDCQ for $x \in D$ is

$$\operatorname{rk}\begin{pmatrix} \partial g_i / \partial x_j (\underline{x}) : i \in B \\ \\ \end{pmatrix} = \# B$$

take only those rows where the constraint is binding at $\underline{x}$

the number of elements in B
= the number of rows in the matrix

Easiest to check NDCQ when we devide in the different cases (according to binding / non-binding).

$\underline{k=2:}$

$\left.\begin{matrix} g_1 = b_1 \\ g_2 = b_2 \end{matrix}\right\}$ $\operatorname{rk}\begin{pmatrix} \partial g_1 / \partial x_j \\ \partial g_2 / \partial x_j \end{pmatrix} = 2$

$\left.\begin{matrix} g_1 = b_1 \\ g_2 < b_2 \end{matrix}\right\}$ $\operatorname{rk}\left(\partial g_1 / \partial x_j \right) = 1$

$\left.\begin{matrix} g_1 < b_1 \\ g_2 = b_2 \end{matrix}\right\}$ $\operatorname{rk}\left(\partial g_2 / \partial x_j \right) = 1$

$\left.\begin{matrix} g_1 < b_1 \\ g_2 < b_2 \end{matrix}\right\}$ no condition

Sufficient condition using concavity in the Kuhn-Tucker case:

If $(\underline{x}^*, \lambda^*)$ satisfies FOC, CSC, C, then we consider

$$\mathcal{L}(\underline{x}, \underline{\lambda}^*) \quad \text{function in } (x_1, \dots, x_n)$$

If $\mathcal{L}(\underline{x}, \lambda^*)$ is concave, then $\underline{x}^*$ is a max.