LECTURE 7

DRE 7007

MATHEMATICS

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SEP 3RD 2012

PLAN:

1. Ordinary differential equations
2. Systems of differential equations
3. Linearizations

Reading:

[FMEA] 5-7
[ME] 24-25

(1) An ODE (ordinary differential equation) is an equation relating a function $y = y(t)$ and its derivative (and possibly higher order derivatives). A first order ODE typically has the form

$$\dot{y} = F(y,t)$$

where F is some function in (y,t). The variable t often is time. An ODE is autonomous if the expression for $\dot{y}$ does not depend on t; i.e that

$$\dot{y} = F(y)$$

in the order one case.

Example: $\dot{y} = ay + b$, with $a, b \in \mathbb{R}$ constants.

Solution methods:
a) Separation
b) Int. factor
c) Linear methods

Constant solution = steady state: $y = \bar{y}$ constant solution

$$\Updownarrow$$

$$a\bar{y} + b = 0$$

$$\Updownarrow$$

$$\bar{y} = -\frac{b}{a} \quad (a \neq 0)$$

Let $z = y - \bar{y}$. Then we have:

$$\underline{z' = y'} \ ; \quad ay + b = a(z + \bar{y}) + b = az + a\left(-\frac{b}{a}\right) + b = az + b - b = \underline{az}$$

Hence

$$y' = ay + b \iff z' = az \quad \text{with } z = y - \bar{y}$$

<u>Solution:</u>

$$z' = az \Rightarrow z = Ce^{at} \Rightarrow y - \bar{y} = Ce^{at} \Rightarrow y = \underline{\bar{y} + Ce^{at}} = -\frac{b}{a} + Ce^{at}$$

<u>Stability:</u>

$a > 0$: $\quad y = -\frac{b}{a} + Ce^{at} \to \infty$ as $t \to \infty$ $\qquad$ <u>not stable</u>

$a < 0$: $\quad y = \bar{y} + Ce^{at} \to \bar{y}$ as $t \to \infty$ $\qquad$ <u>(globally asymptotically) stable</u>

with equilibrium $\bar{y} = -\frac{b}{a}$

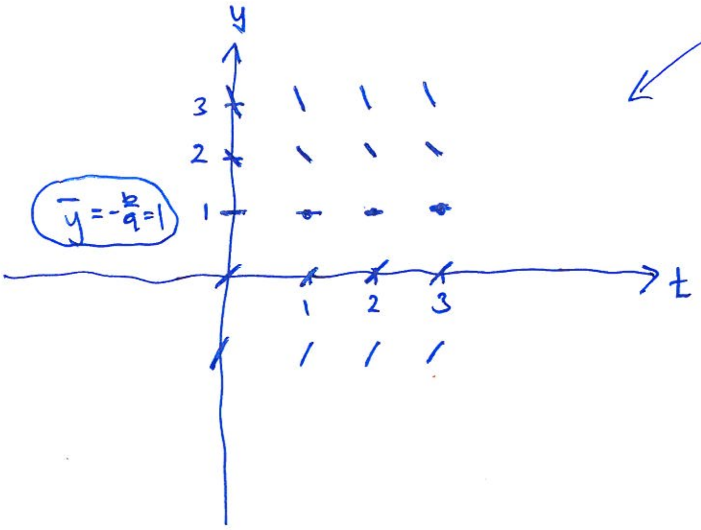
~~$a = 0$: $y = -\frac{b}{a} + C$ constant~~

Note that $y_0 = y(0) = -\frac{b}{a} + C \Rightarrow C = y_0 + \frac{b}{a} = y_0 - \bar{y}$, so C is given by y_0.

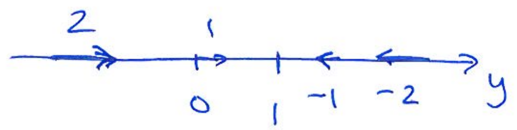
<u>Phase diagram:</u> $\quad$ Case $a = -1, b = 1$

$y = 1, t = 1 \Rightarrow y' = a \cdot y + b = 1 - y$

We draw a small line segment at (t, y) with slope $y' = 1 - y$.



Or



We draw an arrow at (y) with length $|y'| = |1 - y|$ and arrow $\to (+)$ or $\leftarrow (-)$.

(2) Linear systems of ODE's

$$\left.\begin{array}{l} y_1' = a_{11}y_1 + \dots + a_{1n}y_n + b_1 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ y_n' = a_{n1}y_1 + \dots + a_{nn}y_n + b_n \end{array}\right\} \Longleftrightarrow \quad \underline{y}' = A\underline{y} + \underline{b}$$

<u>Steady state</u>: $\overline{\underline{y}} = \begin{pmatrix} \overline{y}_1 \\ \vdots \\ \overline{y}_n \end{pmatrix} \in \mathbb{R}^n$ (constants) such that $A\overline{\underline{y}} + \underline{b} = \underline{0}$

$A\overline{\underline{y}} = -\underline{b}$

(linear system)

If $\overline{\underline{y}}$ is steady state, then $\underline{z} = \underline{y} - \overline{\underline{y}}$ transforms

$\underline{y}' = A\underline{y} + \underline{b}$ into $\underline{z}' = A\underline{z}$.

<u>Thm</u>: If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of A and $\underline{v}_1, \dots, \underline{v}_n$ are corresponding eigenvectors that are linearly independent, then

$\underline{z} = C_1 \cdot \underline{v}_1 \cdot e^{\lambda_1 t} + C_2 \underline{v}_2 e^{\lambda_2 t} + \dots + C_n \underline{v}_n e^{\lambda_n t}$ is gen. solution to $\underline{z}' = A\underline{z}$

In particular, the general solution to the original $\underline{y}' = A\underline{y} + \underline{b}$ is

$$\underline{y} = C_1 \underline{v}_1 e^{\lambda_1 t} + \dots + C_n \underline{v}_n e^{\lambda_n t} + \overline{\underline{y}}$$

<u>Ex</u>:
$$\begin{aligned} y_1' &= y_1 - 3y_2 + 2 \\ y_2' &= 2y_1 - 4y_2 + 2 \end{aligned} \qquad \begin{aligned} y_1 &= 1 + C_1 \cdot 3e^{-t} + C_2 \cdot e^{-2t} \\ y_2 &= 1 + C_1 \cdot 2e^{-t} + C_2 \cdot e^{-2t} \end{aligned}$$

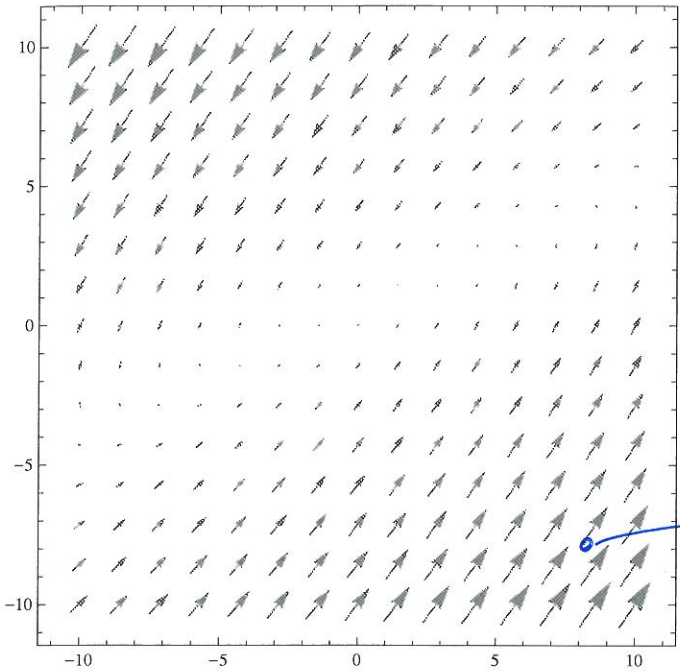
<u>Pf. of thm</u>:

$A\underline{z} = \underline{z}'$ with $\underline{z} = P\underline{u}$ gives diff. eqn. in new var's $\underline{u}$:

$$\left.\begin{array}{l} \underline{z}' = (P\underline{u})' = P\underline{u}' \\ A\underline{z} = AP\underline{u} = PD\underline{u} \end{array}\right\} \begin{array}{l} P\underline{u}' = PD\underline{u} \\ \underline{u}' = D\underline{u} \end{array} \rightarrow \begin{array}{l} u_1' = \lambda_1 u_1 \\ u_2' = \lambda_2 u_2 \\ \vdots \end{array} \rightarrow u_i = C_i e^{\lambda_i t}$$

$$\rightarrow \underline{z} = P\underline{u} = \begin{pmatrix} \cdots \underline{v}_i \cdots \end{pmatrix} \cdot \begin{pmatrix} \vdots \\ C_i e^{\lambda_i t} \\ \vdots \end{pmatrix} = \sum_i \underline{v}_i \cdot C_i e^{\lambda_i t}$$

$\uparrow y_2$



$\rightarrow y_1$

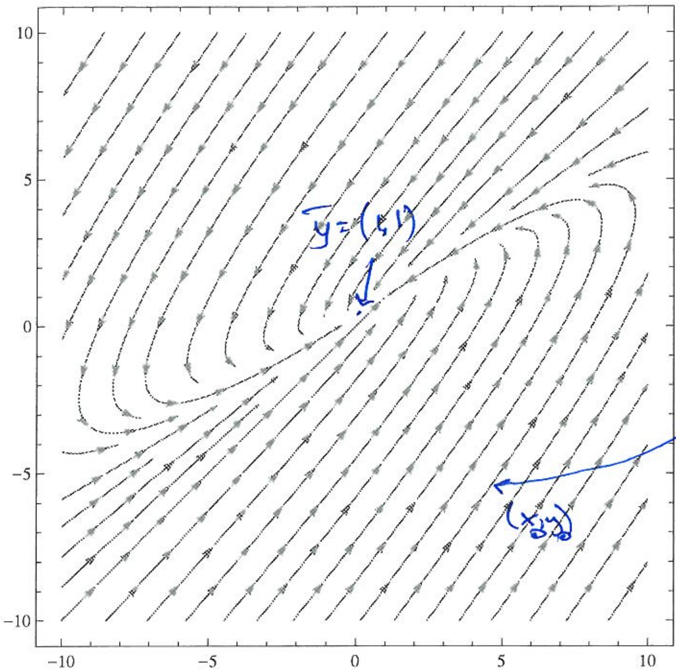
$$\left(y_1 - 3y_2 + 2,\ 2y_1 - 4y_2 + 2\right)$$
$$\parallel$$

vector is (y_1', y_2')

(VectorPlot in Mathematica)

In[5]:= `StreamPlot[{x - 3 y + 2, 2 x - 4 y + 2}, {x, -10, 10}, {y, -10, 10}]` (Mathematica)

Out[5]=



From any starting point (x_0, y_0) at $t=0$, the integral curve tends to steady state $(1,1$

Global asymptotically stable if $y \to \bar{y}$ when $t \to \infty$ for all initial states y_0.

$$\Updownarrow$$

$$\lambda_1 < 0, \lambda_2 < 0, \dots, \lambda_n < 0$$

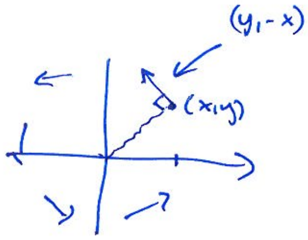
If λ_i are complex eigenvalues, the condition becomes: the real part of λ_i is negative for all i

In the case $n=2$:

$$\lambda_1, \lambda_2 < 0 \iff \begin{cases} \underset{\lambda_1+\lambda_2}{\overset{\shortparallel}{}}\operatorname{tr} A < 0 \\ \underset{\lambda_1 \cdot \lambda_2}{\det A} > 0 \end{cases}$$

$\operatorname{tr} A = \lambda_1 + \lambda_2$, $\det A = \lambda_1 \cdot \lambda_2$

Ex: $y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y$



This characterization also holds for complex eigenvalues

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: $\lambda^2 + 1 = 0$

$$\lambda^2 = \pm\sqrt{-1} = \pm i$$

$$\lambda_1 = \underline{i}, \quad \lambda_2 = \underline{-i}$$

(complex eigenvalues)

Complex numbers: $z = a + ib$, where $a, b \in \mathbb{R}$, "$i = \sqrt{-1}$" (ie $i^2 = -1$)

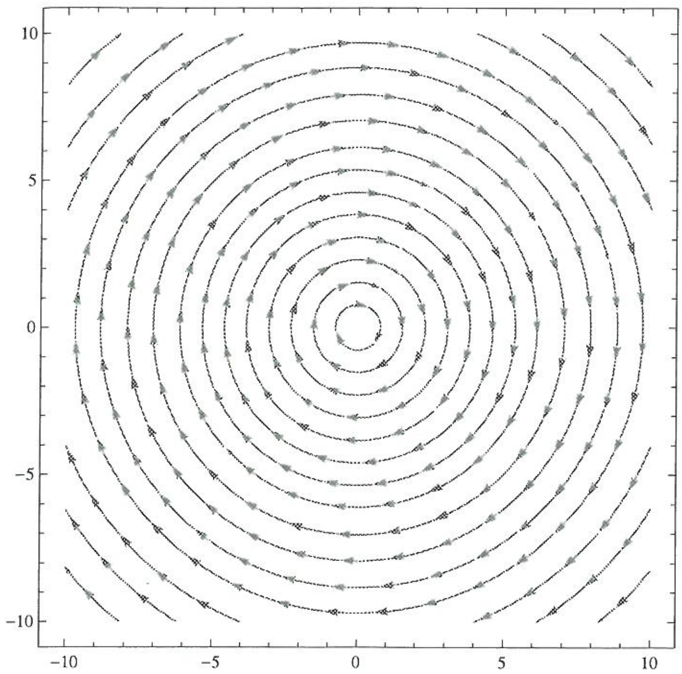
a: real part, b: imaginary part

$z^2 - 2z + 5 = 0$:

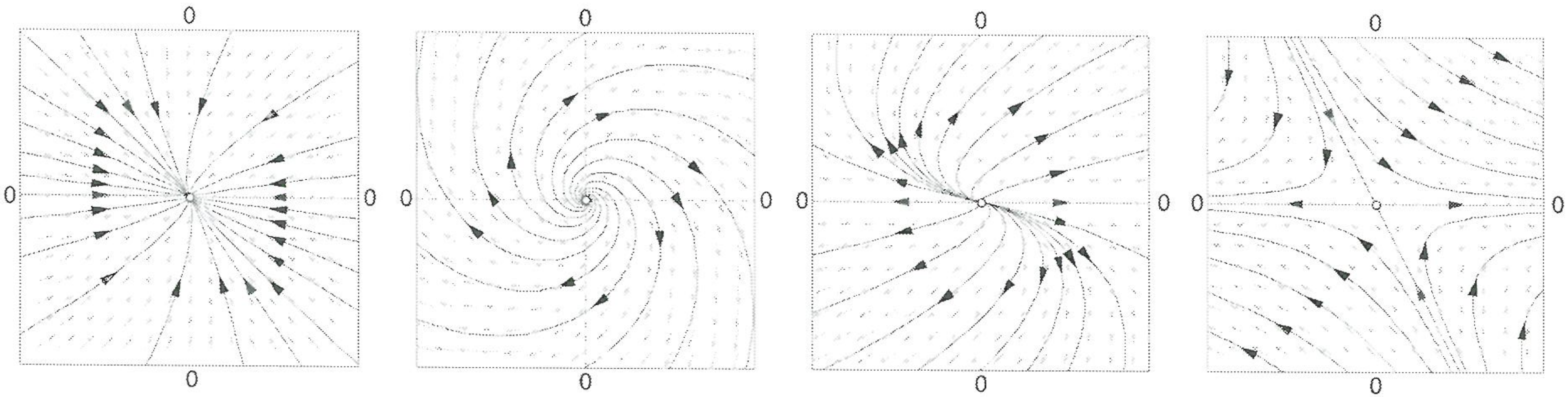
$$z = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm \frac{\sqrt{-16}}{2}$$

$$= 1 \pm \frac{\sqrt{16} \cdot i}{2} = 1 \pm 2i$$

$$z_1 = \underline{1+2i}, \quad z_2 = \underline{1-2i}$$



$y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y$ with pure imaginary eigenvalues
(real part is zero)



Some other examples of vector fields.

③ Linear approximations

$$\left.\begin{aligned} y_1' &= F(y_1, y_2) \\ y_2' &= G(y_1, y_2) \end{aligned}\right\}$$ where F, G are general (non-linear) functions

Steady state: $\underline{y} = \bar{y}$ s.t. $F(\bar{y}) = G(\bar{y}) = 0$.

Linearization:

$$y_1' = F_{y_1}'(\bar{y}) \cdot (y_1 - \bar{y}_1) + F_{y_2}'(\bar{y}) \cdot (y_2 - \bar{y}_2)$$
$$y_2' = G_{y_1}'(\bar{y})(y_1 - \bar{y}_1) + G_{y_2}'(\bar{y}) \cdot (y_2 - \bar{y}_2)$$

$\underline{y}' = A \cdot (\underline{y} - \bar{y})$ or $\boxed{\underline{z}' = A \cdot \underline{z}}$ with $\underline{z} = \underline{y} - \bar{y}$ and

$$A = \begin{pmatrix} F_{y_1}' & F_{y_2}' \\ G_{y_1}' & G_{y_2}' \end{pmatrix}$$

Ex: $x' = x - 3y + 2x^2 + y^2 - xy$
$y' = 2x - y - e^{x+y} + 1$

$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is one steady state (there may be others)

Linearization:

$\underline{z}' = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} \underline{z}$ $\left.\begin{aligned} \det A &= -2 + 3 = 1 > 0 \\ \operatorname{tr} A &= 1 + (-2) = -1 < 0 \end{aligned}\right\}$ Globally asymptotically stable at $(0,0)$.