LECTURE 8

DKE 7007

MATHEMATICS

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SEP 4TH 2017

PLAN:

1. Optimal control theory - continuous case
   - Pontryagin's maximum principle
   - Sufficient conditions

Reading

[FMEA] 9 (8.10)

Remember: Thu and Fri 14-16 at C2-095

1.

Problem:

$$\max \int_{t_0}^{t_1} f(t,x,u)\,dt \quad \text{such that} \begin{cases} x(t_0) = x_0 \\ \dot{x} = g(t,x,u) \\ u \in U \subseteq \mathbb{R} \\ \text{Either: } \begin{matrix} \text{a) } x(t_1) = x_1 \\ \text{b) } x(t_1) \text{ is free} \end{matrix} \end{cases}$$

$u = u(t)$ control variable $\quad u: [t_0,t_1] \to U \quad$ ($U \subseteq \mathbb{R}$ control region)

$x = x(t)$ state variable $\quad x: [t_0,t_1] \to \mathbb{R}$

When u is chosen, then x is given by the ODE $\dot{x} = g(t,x,u)$ and initial condition $x(t_0) = x_0$. We may or may not have an additional condition on $x(t_1)$.

Hence, for each u, we may compute $\int_{t_0}^{t_1} f(t,x,u)\,dt$, and we want to find u such that this is maximal.

Necessary conditions:

The Hamiltonian $H(t,x,u,p) = p_0 f(t,x,u) + p \cdot g(t,x,u)$
where $p_0 \in \mathbb{R}$ and $p: [t_0,t_1] \to \mathbb{R}$ is a function in t.

Maximum principle: (Pontryagin)

If x^*, u^* is an optimal pair, then there is a function $p(t)$ and a number p_0 such that $(p_0, p(t)) \neq (0,0)$ for all $t \in [t_0,t_1]$, such that

(A) $u \mapsto H(t,x^*,u,p)$ has a maximum at $u = u^*$

(B) $\dot{p}(t) = -H'_x(t,x^*,u^*,p)$

(C) Transversality: $\begin{cases} \text{a) } \underline{x(t_1)=x_1}: \text{ no condition} \\ \text{b) } \underline{x(t_1) = \text{free}}: \; p(t_1) = 0 \end{cases}$

* We may assume $p_0 = 0$ or $p_0 = 1$. If $x(t_1)$ is free, then we may assume $p_0 = 1$.

Sufficient conditions:

Theorem: (Mangasarian)
Suppose that (x^*, u^*) is admissible and satisfies (A) - (C) above with $p_0 = 1$.
If U is convex and $(x,u) \mapsto H(t,x,u,p)$ is concave for all $t \in [t_0,t_1]$, then (x^*, u^*) is an optimal pair.

Ex I: $\max \int_0^T (1 - tx - u^2)\,dt$, $\quad \dot{x} = u$, $x(0) = x_0$, $U = \mathbb{R}$ $\quad (x_0, T \text{ given})$

Necessary condition:

$H = p_0 \cdot (1 - tx - u^2) + p \cdot u$

$p_0 = 0$: $p(t) \neq 0$

(B) $\dot{p} = 0 \Rightarrow p(t) = c \neq 0$ constant

$H = pu = c \cdot u$ has no max as a fn. in u, so (A) not satisfied

$\Downarrow$

no solutions with $p_0 = 0$

$p_0 = 1$: $H = 1 - tx - u^2 + pu$

$\dot{x} = u$

$x(0) = x_0$

(A) $\dfrac{\partial H}{\partial u} = p - 2u = 0 \Rightarrow u = \frac{1}{2}p$

(B) $p' = -(-t) = t \Rightarrow p = \frac{1}{2}t^2 + C$

(C) $p(T) = 0 \Rightarrow C = -\frac{1}{2}T^2 \Rightarrow p(t) = \underline{\frac{1}{2}t^2 - \frac{T^2}{2}} \Rightarrow u = \underline{\frac{1}{4}t^2 - \frac{T^2}{4}}$

$\dot{x} = u = \frac{1}{4}t^2 - \frac{T^2}{4} = \frac{1}{12}t^3 - \frac{T^2}{4}t + C$

$x(0) = x_0 \Rightarrow C = x_0 \Rightarrow x = \underline{\frac{1}{12}t^3 - \frac{T^2}{4}t + x_0}$

$U = \mathbb{R}$ convex & $H = 1 - tx - u^2 + pu$ concave in (x, u) $\Rightarrow$ $u = \frac{1}{4}t^2 - T^2/4$, $x = \frac{1}{12}t^3 - \frac{T^2}{4}t + x$ is the maximizer