# LECTURE 9

# DRE 7007

MATHEMATICS

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Reading:

[FMEA] 12

[S] 11-12

PLAN:

1. Optimal control theory - discrete time, finite horizon
2. Infinite horizon

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1. **Problem** (finite horizon)

$$\max_{(\min)} \sum_{t=0}^{T} f(t, x_t, u_t) \quad \text{subject to} \quad \begin{cases} x_0 \text{ given} \\ x_{t+1} = g(t, x_t, u_t) \quad \text{for } t = 0, 1, \ldots, T-1 \\ u_t \in U \subseteq \mathbb{R} \end{cases}$$

Comments:

- the set $U \subseteq \mathbb{R}$ of admissible controls may depend on the state variable $x_t$
- Sometimes $f(T, x_T, u_T) = S(x_T)$ and is called a scrap value function.

A) Dynamic programming and the Bellman equation

Let $s \leq T$ and define the optimal value function

$$J_s(x) = \max_{(u_s, \ldots, u_T)} \sum_{t=s}^{T} f(t, x_t, u_t) \quad \text{subject to} \quad \begin{cases} x_s = x \\ x_{t+1} = g(t, x_t, u_t) \quad \text{for } t \geq s \\ u_t \in U \end{cases}$$

Bellman equation:

$$J_s(x) = \max_{u \in U} \left\{ f(s, x, u) + J_{s+1}(g(s, x, u)) \right\} \quad \text{for } s = 0, 1, 2, \ldots, T-1$$

$$J_T(x) = \max_{u \in U} f(T, x, u)$$

Ex T:

$$\max \sum_{t=0}^{3} (1+x_t - u_t^2) \qquad x_{t+1} = x_t + u_t \,, \quad x_0 = 0 \,, \quad U = \mathbb{R}$$

$$J_3(x) = \max_{u \in \mathbb{R}} \left(1+x-u^2\right) = 1+x \quad \text{at } u_3 = 0$$

$$J_2(x) = \max_{u_2 \in \mathbb{R}} \left(1+x-u_2^2\right) + J_3(x+u_2) = \max_u \left\{1+x-u^2+1+x+u\right\}$$

$' = -2u+1 \qquad u = 1/2$

$'' = -2 \qquad$ ok.

$$= 2+2x+1/4 = \underline{9/4+2x} \,, \quad u_2 = 1/2$$

$$J_1(x) = \max \quad 1+x-u^2 + J_2(x+u) = \max \quad 1+x-u^2+9/4+2x+2u$$

$' = -2u+2 \qquad u = 1$

$' = -2$

$$= 1+x-1+9/4+2x+2$$

$$= \underline{17/4+3x} \,, \quad u_1 = 1$$

$$J_0(x) = \max \quad 1+x-u^2 + 17/4 + 3(x+u)$$

$' = -2u+3 = 0 \qquad u = 3/2$

$' = -2$

$$= 1+x-9/4+17/4+3x+9/2 = \underline{4x+15/2} \qquad u_0 = 3/2$$

<u>With $x_0 = 0$:</u>

$u_0 = 3/2 \qquad x_0 = 0 \qquad J_0(x) = \underline{4x+15/2} = \underline{\underline{15/2}}$

$u_1 = 1 \qquad x_1 = 3/2$

$u_2 = 1/2 \qquad x_2 = 5/2$

$u_3 = 0 \qquad x_3 = 3$

## Alternative solution methods

### B) Euler equation

$$\max \sum_{t=0}^{T} F(t, x_t, x_{t+1}) \quad \text{subject to} \quad \begin{cases} x_0 \text{ given} \\ x_1, \dots, x_{T+1} \in U = \mathbb{R} \text{ free} \end{cases}$$

- special case: $x_{t+1} = u_t$, $g(t, x_t, u_t) = u_t$, $U = \mathbb{R}$
- many problems are covered by this via transformations

Euler equation:

If $x_0^*, x_1^*, \dots, x_{T+1}^*$ is optimal solution, then it satisfies the difference equations

$$\boxed{\begin{aligned} F_2'(t, x_t, x_{t+1}) + F_3'(t-1, x_{t-1}, x_t) &= 0 \qquad \text{for } t = 1, 2, \dots, T \\ F_3'(t-1, x_{t-1}, x_t) &= 0 \qquad t = T+1 \end{aligned}}$$

where $F_2', F_3'$ are the partial derivatives of $F$ with respect to variable 2 and 3.

Ex: $$\max \sum_{t=0}^{T-1} \ln(x_t - \beta x_{t+1}) + \ln x_T$$

$$\begin{cases} F_2'(t, x_t, x_{t+1}) + F_3'(t-1, x_{t-1}, x_t) = 0 \qquad t = 1, \dots, T \\ \dfrac{1}{x_t - \beta x_{t+1}} - \dfrac{\beta}{x_{t-1} - \beta x_t} = 0 \\ x_{t-1} - \beta x_t = \beta(x_t - \beta x_{t+1}) \\ \underline{\beta^2 x_{t+1} - 2\beta x_t + x_{t-1} = 0} \qquad \text{(second order difference eqn.)} \end{cases}$$

$$\begin{cases} F_3'(T, x_T, x_{T+1}) = 0 \\ 0 = 0 \\ \text{(no condition)} \end{cases} \qquad (\text{since } F(T, x_T, x_{T+1}) = \ln(x_T))$$

## C) Maximum principle and Hamiltonian

$$\max \sum_{t=0}^{T} f(t,x_t,u_t) \quad \text{s.t.} \begin{cases} x_0 \text{ given} \\ x_{t+1} = g(t,x_t,u_t) \\ x_T \text{ free} \\ u_t \in U \subseteq \mathbb{R} \end{cases}$$

**Hamiltonian:** $$H = \begin{cases} f(t,x,u) + p\,g(t,x,u) & , t<T \\ f(t,x,u) & , t=T \end{cases}$$

**Maximum principle** (necessary conditions)

If $(x_t^*, u_t^*)$ is optimal, then there is a sequence $p_t$, $t=0,1,\dots,T$ with

A) $H'_u(t,x_t^*,u_t^*,p_t)\cdot(u-u_t^*) \le 0$ for all $u \in U$

B) $p_{t-1} = H'_x(t,x_t^*,u_t^*,p_t)$ , $t=1,2,\dots,T$

C) $p_T = 0$

**Sufficient conditions**

Suppose that $(x_t^*, u_t^*)$ satisfy the conditions above, and that in addition

$H(t,x,u,p)$ is concave in $(x,u)$ for all $t$

Then $(x_t^*, u_t^*)$ is optimal.

## ② Infinite Horizon dynamic programming

$$\max \sum_{t=0}^{\infty} \beta^t f(x_t, u_t) \quad \text{subject to} \quad \begin{cases} x_0 \text{ given} \\ x_{t+1} = g(x_t, u_t) \\ u_t \in U \subseteq \mathbb{R} \end{cases}$$

**Note:**

* instead of $t \in T$ we have infinite horizon $t \to \infty$
* $\beta^t f(x_t, u_t) = f(t, x_t, u_t)$ ; $g(x_t, u_t) = g(t, x_t, u_t)$
  ($0 < \beta < 1$ is the one-periode discount factor)
* we assume that $f(x_t, u_t)$ is bounded, i.e. $|f(x_t, u_t)| < M$ for all $t$ for some number $M > 0$. This implies that the sum
  $$\sum_{t=0}^{\infty} \beta^t f(x_t, u_t) \leq \sum_{t=0}^{\infty} \beta^t M = \frac{M}{1-\beta} \quad \text{is finite}$$

**Bellman equation:** Let $J(x) = J_0(x) = \max_{(u_t)} \sum_{t=0}^{\infty} \beta^t f(x_t, u_t)$ subj. to $\ldots \begin{cases} x_0 = x \\ \vdots \end{cases}$

$$\boxed{J(x) = \max_{u \in U} \left\{ f(x, u) + \beta J(g(x, u)) \right\}}$$

- Functional equations: we want to solve for the function $J(x)$. Difficult to find max when $J(x)$ is not known, hence difficult to find $J(x)$ by solving the Bellman equation.

- When $0 < \beta < 1$ and $|f(x_t, u_t)| < M$, the Bellman equation has a unique bounded solution $J^*(x)$. If we "guess" $\tilde{J}(x)$ and it fits in the equation, it is therefore the unique solution.

Ex:
Problem 12.3.1 in [FMEA]. See also (SM) Student Manual (online)