Solutions Problem Sheet I:

1. a) $\begin{vmatrix} 2-\lambda & -3 \\ 7 & -8-\lambda \end{vmatrix} = \lambda^2+6\lambda+5=0$

$\underline{\underline{\lambda_1=-1,\ \lambda_2=-5}}$

$\underline{\lambda=-1:}$

$\begin{pmatrix} 3 & -3 \\ 7 & -7 \end{pmatrix}\underline{x}=\underline{0} \Rightarrow \underline{x} = \underline{\underline{t\cdot\begin{pmatrix}1\\1\end{pmatrix}}}$

$\underline{\lambda=-5:}$

$\begin{pmatrix} 7 & -3 \\ 7 & -3 \end{pmatrix}\underline{x}=\underline{0} \Rightarrow \underline{x} = \underline{\underline{t\cdot\begin{pmatrix}3\\7\end{pmatrix}}}$

b) $\begin{vmatrix} 1-\lambda & 3 & 0 \\ 2 & -\lambda & 0 \\ 1 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)\cdot(\lambda^2-\lambda-6)=0$

$\underline{\underline{\lambda_1=2,\ \lambda_2=3,\ \lambda_3=-2}}$

$\underline{\lambda=2:}$

$\begin{pmatrix} -1 & 3 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{pmatrix}\underline{x}=\underline{0}$

$\downarrow$

$\begin{pmatrix} -1 & 3 & 0 \\ 0 & 4 & 0 \\ 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 3 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\underline{x}=\underline{0} \Rightarrow \underline{x} = \underline{\underline{t\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}}$

$\underline{\lambda=3:}$

$\begin{pmatrix} -2 & 3 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & -1 \end{pmatrix}\underline{x}=\underline{0}$

$\downarrow$

$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}\underline{x}=\underline{0} \Rightarrow \underline{x} = \underline{\underline{t\begin{pmatrix}3\\2\\1\end{pmatrix}}}$

$\underline{\lambda=-2:}$

$\begin{pmatrix} 3 & 3 & 0 \\ 2 & 2 & 0 \\ 1 & -1 & 4 \end{pmatrix}\underline{x}=\underline{0}$

$\downarrow$

$\begin{pmatrix} 1 & -1 & 4 \\ 0 & 4 & -8 \\ 0 & 0 & 0 \end{pmatrix}\underline{x}=\underline{0} \Rightarrow \underline{x} = \underline{\underline{t\cdot\begin{pmatrix}-2\\2\\1\end{pmatrix}}}$

c) $\begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0$

$\underline{\lambda_1 = \lambda_2 = 3}$

$\lambda = 3$: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \underline{x} = \underline{0} \Rightarrow \underline{x} = t \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2. a) $|A| = (-1)\cdot(-5) = \underline{5} \neq 0 \Rightarrow \text{rk } A = \underline{2}$

$|A| = 2\cdot 3 \cdot (-2) = \underline{-12} \neq 0 \Rightarrow \text{rk } A = \underline{3}$

$|A| = 3 \cdot 3 = \underline{9} \neq 0 \Rightarrow \text{rk } A = \underline{2}$

b) $\lambda_1, \lambda_2 < 0 \Rightarrow$ neg. defn.

$\lambda_1, \lambda_2 > 0$, $\lambda_3 < 0 \Rightarrow$ indefn.

$\lambda_1 = \lambda_2 > 0 \Rightarrow$ pos. defn.

Alt. defn: A pos. defn etc. $\Leftrightarrow$ $\underline{x}^T A \underline{x}$ is

This gives

$\to \begin{pmatrix} 2 & 2 \\ 2 & -8 \end{pmatrix}$ indef.

$\begin{pmatrix} 1 & 2.5 & 0.5 \\ 2.5 & 0 & -0.5 \\ 0.5 & -0.5 & 2 \end{pmatrix}$ indef.

$\begin{pmatrix} 3 & 0.5 \\ 0.5 & 3 \end{pmatrix}$ pos. defn.

c) i) Diagonalizable: $D = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$ $P = \begin{pmatrix} 1 & 3 \\ 1 & 7 \end{pmatrix}$

ii) Diagonalizable: $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ $P = \begin{pmatrix} 0 & 3 & -2 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

iii) Not diag.: $\lambda_1 = \lambda_3 = 3$ (mult. 2)

Eigenvect: $\lambda \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ only one lin. indep. eigenvector.

3. a) $D_1 = 2 > 0$

$D_2 = 8 - 1 = 7 > 0$

$D_3 = |A| = 6 > 0$

$\Rightarrow$ A pos. defn.

b) $D_1 = 1 > 0$

$D_2 = 1 - 4 = -3 < 0 \Rightarrow$ A indefn.

4.

a) $\underline{x}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$ gives $\underline{x}_{t+1} = A \cdot \underline{x}_t$, $A = \begin{pmatrix} 0.75 & 0.35 \\ 0.25 & 0.65 \end{pmatrix}$

$\underline{x}_t = A^t \cdot \underline{x}_0$ for all t

Alt I: Eigenvalue/-vector for A

$$\begin{vmatrix} 0.75-\lambda & 0.35 \\ 0.25 & 0.65-\lambda \end{vmatrix} = \lambda^2 - 1.4\lambda + 0.4 = 0$$

$$\underline{\lambda_1 = 1}, \ \underline{\lambda_2 = 0.4}$$

$\underline{\lambda = 1}$: $\begin{pmatrix} -0.25 & 0.35 \\ 0.25 & -0.35 \end{pmatrix} \underline{x} = \underline{0} \Rightarrow \underline{x} = t \cdot \underline{\begin{pmatrix} 7 \\ 5 \end{pmatrix}}$

$\underline{\lambda = 0.4}$: $\begin{pmatrix} 0.35 & 0.35 \\ 0.25 & 0.25 \end{pmatrix} \underline{x} = \underline{0} \Rightarrow \underline{x} = t \cdot \underline{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

$$A^t = (PDP^{-1})^t = PD^tP^{-1} = \begin{pmatrix} 7 & 1 \\ 5 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1^t & 0 \\ 0 & 0.4^t \end{pmatrix} \cdot \frac{1}{(-12)} \begin{pmatrix} -1 & -1 \\ -5 & 7 \end{pmatrix}$$

$\downarrow$ as $t \to \infty$

$$\begin{pmatrix} 7 & 1 \\ 5 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \frac{1}{-12} \begin{pmatrix} -1 & -1 \\ -5 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 0 \\ 5 & 0 \end{pmatrix} \frac{1}{(-12)} \cdot \begin{pmatrix} -1 & -1 \\ -5 & 7 \end{pmatrix} = \frac{1}{-12} \cdot \begin{pmatrix} -7 & -7 \\ -5 & -5 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 7 & 7 \\ 5 & 5 \end{pmatrix}$$

$$\underline{x}_t = A^t \underline{x}_0 \to \frac{1}{12} \begin{pmatrix} 7 & 7 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 7x_0 + 7y_0 \\ 5x_0 + 5y_0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 7/12 \\ 5/12 \end{pmatrix}}}$$

Alt. 2:

$$x_0 = c_1 \cdot \begin{pmatrix} 7 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow A^t x_0 = c_1 A^t \begin{pmatrix} 7 \\ 5 \end{pmatrix} + c_2 A^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= c_1 \cdot 1^t \cdot \begin{pmatrix} 7 \\ 5 \end{pmatrix} + c_2 \cdot (0.4)^t \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\to \underline{c_1 \cdot \begin{pmatrix} 7 \\ 5 \end{pmatrix}}$$

Since $x_0 + y_0 = 1$, we have

$$c_1 \cdot 7 + c_1 \cdot 5 = 1 \Rightarrow c_1 = 1/12 \Rightarrow A^t x_0 \to \underline{\underline{\begin{pmatrix} 7/12 \\ 5/12 \end{pmatrix}}}$$