Problem Sheet 10
DRE 7007 Mathematics

BI Norwegian Business School

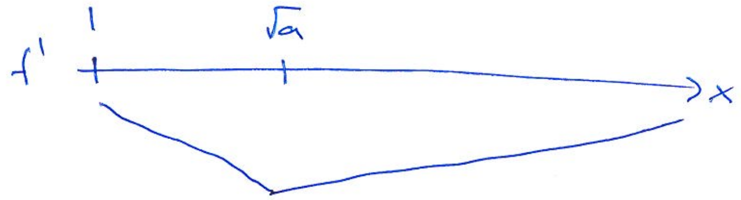
Solutions Problem Sheet 10

1. $f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$, $x > 1$ $\qquad D = \{x \in \mathbb{R} : x > 1\} = (1, \infty)$

a) Assume that $a \in (1,3)$. Then

$$f'(x) = \frac{1}{2}\left(1 + a \cdot \left(-\frac{1}{x^2}\right)\right) = \frac{1}{2}\left(1 - \frac{a}{x^2}\right) \Rightarrow f'(x) = 0 \text{ for } x = \sqrt{a}.$$

Hence the sign of f' is given by



The minimal value of $f(x)$ is at $x = \sqrt{a}$ with $f(\sqrt{a}) = \frac{1}{2}\left(\sqrt{a} + \frac{a}{\sqrt{a}}\right)$
$= \sqrt{a} > 1$, Hence $f : D \to D$.

b) If $1 < a < 3$, then f is a contraction, since we have for $x, x' > 1$ that

$$|f(x) - f(x')| = \left|\frac{1}{2}\left(x + \frac{a}{x}\right) - \frac{1}{2}\left(x' + \frac{a}{x'}\right)\right| = \frac{1}{2}\left|x - x' + \frac{ax' - ax}{xx'}\right|$$
$$= \frac{1}{2}|x - x'| \cdot \left|1 - \frac{a}{xx'}\right| < |x - x'| \cdot K$$

where $K = \sup \frac{1}{2}\left|1 - \frac{a}{xx'}\right| < 1$ (when $xx' \to \infty$ then $\left|1 - \frac{a}{xx'}\right| \to 1$ and when $xx' \to 1$ then $\left|1 - \frac{a}{xx'}\right| \to |1 - a| < 2$).

The fixed point is given by
$$\frac{1}{2}\left(x + \frac{a}{x}\right) = x$$
$$\frac{1}{2}\left(-x + \frac{a}{x}\right) = 0$$
$$x^2 = a$$
$$\underline{\underline{x = \sqrt{a}}}$$

For $\underline{a = 1}$: $\frac{1}{2}\left(x + \frac{1}{x}\right) = x$ has fixed point
$x = \sqrt{1} = 1$ but $1 \notin D$. <u>No fixed pt. on D</u>.

$\underline{a = 3}$: $\frac{1}{2}\left(x + \frac{3}{x}\right) = x$

$$\frac{1}{2}\left(-x + \frac{3}{x}\right) = 0$$
$$x^2 = 3$$
$$\underline{x = \sqrt{3}}$$

<u>$x = \sqrt{3}$ is fixed pt</u> (even if f is not nec. a contraction)

c) D is not complete. The Cauchy series $x_n = 1 + 1/n$ has no limit in D.

2. $$F(x) = \begin{cases} \{2\} & ; \ x \in [0,1) \\ \{0,2\} & ; \ x = 1 \\ \{0\} & , \ x \in (1,2] \end{cases}$$

$\text{Graph}(F) = \{(x,2) : x \in [0,1]\} \cup \{(x,0) : x \in [1,2]\}$

Consider $F : [0,2] \rightrightarrows [0,2]$. It is

- upper hemicont. since Graph(F) is compact.
- F(x) non-empty, but not always convex set.

$$F(1) = \{0,2\} \text{ not convex}$$

- K is compact, non-empty convex.

Conclusion: Kakutani's thm. does not apply.

Fixed pts:

a) $x \in [0,1)$: $F(x) = \{2\}$, $x \notin F(x)$
b) $x = 1$: $F(1) = \{0,2\}$ $1 \notin F(1)$
c) $x \in (1,2]$: $F(x) = \{0\}$ $x \notin F(x)$

no fixed points