BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

DRE7017.

1.)

a) $\begin{pmatrix} 2 & 1 & 5 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \underline{x} = \begin{pmatrix} 9 \\ 3 \\ 2 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 2 & 1 & 5 & 9 \\ 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 3 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 4 & 4 \end{array}\right) \qquad \begin{array}{l} x = 1 \\ y = 2 \\ z = 1 \end{array}$$

b) $\underline{z} = \begin{pmatrix} x-1 \\ y-2 \\ z-1 \end{pmatrix}$: $\qquad \underline{z}' = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \underline{z}$

$$\begin{vmatrix} 2-\lambda & 1 & 5 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\left((2-\lambda)(1-\lambda)-1\right) + 5\,(-1)(1-\lambda)$$

$$= (1-\lambda)\cdot(\lambda^2 - 3\lambda - 4) = 0$$

$\underline{\lambda = 1} \quad \underline{\lambda = 4} \quad \underline{\lambda = -1}$

$\underline{\lambda = 1}$: $\begin{pmatrix} 1 & 1 & 5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{v_1} = \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}$

$\underline{\lambda = 4}$: $\begin{pmatrix} -2 & 1 & 5 \\ 1 & -3 & 0 \\ 1 & 0 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{v_2} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

$\underline{\lambda = -1}$: $\begin{pmatrix} 3 & 1 & 5 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{v_3} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

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$$\underline{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + C_1 \cdot \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix} e^{t} + C_2 \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} e^{4t} + C_3 \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} e^{-t}$$

c) $t \to \infty$: $e^{t}, e^{4t} \to \infty$, $e^{-t} \to 0$

$\underline{C_1 = C_2 = 0}$

$$\underline{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + C_3 \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} e^{-t}$$

$$\to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

as $t \to \infty$

$\underline{C_1 = C_2 = 0:}$

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + C_1 \cdot \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix} + C_2 \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + C_3 \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + C_3 \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - 2C_3 \\ 2 + C_3 \\ 1 + C_3 \end{pmatrix}$$

$$\begin{aligned} x_0 &= 1 - 2C_3 \\ y_0 &= 2 + C_3 \\ z_0 &= 1 + C_3 \end{aligned}$$

(parametric)

$\Leftrightarrow$ $\underline{C_3 = z_0 - 1:}$

$$\begin{aligned} x_0 &= 1 - 2(z_0 - 1) \\ &= 3 - 2z_0 \\ y_0 &= 2 + (z_0 - 1) \\ &= 1 + z_0 \end{aligned}$$

$$\underline{\{(x_0, y_0, z_0) : x_0 = 3 - 2z_0,\ y_0 = 1 + z_0\}}$$

(equations)

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2.) $h = xw - yz - a(x^2+4y^2) - b(4z^2+9w^2) \quad (a, b > 0)$

a)

$$\left.\begin{aligned} h'_x &= w - 2ax \\ h'_y &= -z - 8ay \\ h'_z &= -y - 8bz \end{aligned}\right\} \quad H = \begin{pmatrix} -2a & 0 & 0 & 1 \\ 0 & -8a & -1 & 0 \\ 0 & -1 & -8b & 0 \\ 1 & 0 & 0 & -18b \end{pmatrix}$$

$h'_w = x - 18bw$

ok $D_1 = -2a < 0$

ok. $D_2 = 16a^2 > 0$

Necessary condition for concavity:

$D_3 = -2a(64ab-1)$

$D_4 = 36ab(64ab-1) - 1\cdot(64ab-1)$

$= (36ab-1)(64ab-1)$

$64ab - 1 \geq 0 \quad (D_3 \leq 0)$

$36ab - 1 \geq 0 \quad (D_4 \geq 0)$

$\Updownarrow$

$ab \geq \frac{1}{36} \quad \left(\geq \frac{1}{64}\right)$

If $ab = \frac{1}{36}$:

$D_1 = -2a < 0$

$D_2 = 16a^2 > 0$

$D_3 = -2a(64ab-1) < 0$

$D_4 = 0$

$\Delta_1 = -2a, -8a, -8b, -18b \leq 0$

$\Delta_2 = 16a^2, 16ab, 36ab-1, 64ab-1, 144ab, 144b^2 \geq 0$

$\Delta_3 =$
$-2a(64ab-1), -18b(64ab-1), -8a(36ab-1), -8b(36ab-1) \leq 0$

$\Delta_4 = D_4 = 0$

ok.

h concave $\Leftrightarrow$ $\underline{\underline{ab \geq 1/36}}$

h is never convex

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b) $\mathcal{L} = xw - yz - \lambda_1(x^2+4y^2) - \lambda_2(4z^2+9w^2)$ $= h$ with $a=\lambda_1$, $b=\lambda_2$

FOC
$$\begin{cases} \mathcal{L}'_x = w - 2\lambda_1 x = 0 \\ \mathcal{L}'_y = -z - 8\lambda_1 y = 0 \\ \mathcal{L}'_z = -y - 8\lambda_2 z = 0 \\ \mathcal{L}'_w = x - 18\lambda_2 w = 0 \end{cases}$$

(1),(4): $w = 2\lambda_1 x = 36\lambda_1\lambda_2 w$

$w=0, x=0$ or $\lambda_1\lambda_2 = 1/36$

(2),(3): $z = -8\lambda_1 y = 64\lambda_1\lambda_2 z$

$z=0, y=0$ or $\lambda_1\lambda_2 = 1/64$

C
$$\begin{cases} x^2+4y^2=4 \\ 4z^2+9w^2=36 \end{cases}$$

FOC + C:

(1) $x=0, w=0$: $y^2=1$, $z^2=9$

$\lambda_1\lambda_2 = 1/64$

$(0,1,3,0)$, $(0,-1,-3,0)$: $\lambda_1 = -3/8$, $\lambda_2 = -1/24$, $f=-3$

$(0,-1,3,0)$, $(0,1,-3,0)$: $\lambda_1 = 3/8$, $\lambda_2 = 1/24$, $f=3$

(2) $y=0, z=0$: $x^2=4$, $w^2=4$

$\lambda_1\lambda_2 = 1/36$

$(2,0,0,2)$, $(-2,0,0,-2)$: $\lambda_1 = 1/2$, $\lambda_2 = 1/18$, $f=4$

$(2,0,0,-2)$, $(-2,0,0,2)$: $\lambda_1 = -1/2$, $\lambda_2 = -1/18$, $f=-4$

Cond. for max:

$(2,0,0,2)$, $(-2,0,0,-2)$: $f=4$

$\mathcal{L} = \mathcal{L}(x,y,z,w;\ 1/2, 1/18)$ concave since $ab = \lambda_1\lambda_2 = 1/36$ fr a).

$\Downarrow$

$\mathbf{x} = (\pm 2, 0, 0, \pm 2)$, $f=4$ is global max

3) $\max \sum_{t=0}^{4} 1+x_t^2-u_t^2$ when $\begin{cases} x_0=1 \\ x_{t+1}=x_t+u_t \\ u_t\in U=[0,1] \end{cases}$

$$J_t(x)=\max_{u\in U}\left\{1+x^2-u^2+J_{t+1}(x+u)\right\}$$

a) $J_4=\max_{u\in U} 1+x^2-u^2 = \underline{1+x^2}$ $\quad(\underline{u=0})$

$$J_3=\max_{u\in U}\underbrace{1+x^2-u^2+1+(x+u)^2}_{2+2x^2+2xu}=\underline{2+2x+2x^2}\quad(\underline{u=1})$$

$x_t\geq 1$

Since $u_t\geq 0$, $x_{t+1}=x_t+u_t\geq x_t$
and $x_t\geq x_0=1$
for all t

$$J_2=\max_{u\in U}\underbrace{1+x^2-u^2+2+2(x+u)+2(x+u)^2}_{3+2x+3x^2+u^2+2u+4xu}=\underline{6+6x+3x^2}\quad(u=1)$$

$$J_1=\max_{u\in U}\underbrace{1+x^2-u^2+6+6(x+u)+3(x+u)^2}_{7+6x+4x^2+2u^2+6u+6xu}=\underline{15+12x+4x^2}\quad(u=1)$$

$$J_0=\max_{u\in U}\underbrace{1+x^2-u^2+15+12(x+u)+4(x+u)^2}_{16+12x+5x^2+3u^2+12u+8xu}=\underline{31+20x+5x^2}\quad(u=1)$$

Max value: $J_0(x_0)=31+20\cdot 1+5\cdot 1^2=\underline{\underline{56}}$

$x_0=1+h$ $(h>0)$: $J_0(x_0)=31+20(1+h)+5(1+h)^2$
$=56+30h+5h^2>56$

max value will increase with increasing x_0

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4)

a) $f_1 = x(1-x) = x - x^2 \qquad f_1'(x) = 1-2x \quad f_1''(x) = -2$

$$\|f_1\| = \sup_{x\in[0,1]} x - x^2 = f_1(1/2) = \tfrac{1}{2} - \tfrac{1}{4} = \underline{\underline{1/4}}$$

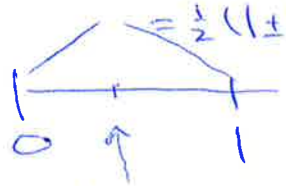
$$f_2 = 2x(1-x)^2 = 2x(1-2x+x^2) \qquad f_2' = 2-8x+6x^2$$
$$= 2x - 4x^2 + 2x^3 \qquad = 6\cdot(x-1)(x-1/3)$$

$$\|f_2\| = f_2(1/3) = \tfrac{2}{3}\cdot\left(\tfrac{2}{3}\right)^2 = \underline{\underline{8/27}}$$

b) $$f_1 - f_2 = (x-x^2) - (2x-4x^2+2x^3)$$
$$= -x + 3x^2 - 2x^3 = h(x)$$

$$h' = -1+6x-6x^2 = 0$$
$$x = \frac{-6 \pm \sqrt{36-4\cdot 6}}{-12} = \frac{1}{2} \pm \frac{\sqrt{12}}{12} = \frac{1}{2} \pm \frac{2\sqrt{3}}{12} = \frac{1}{2}\left(1 \pm \frac{\sqrt{3}}{3}\right) = \frac{1}{2}(1 \pm 1/\sqrt{3})$$

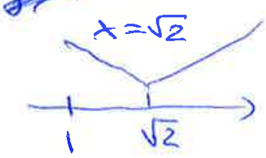
$$d(f_1,f_2) = \sup_{x\in[0,1]} \big(f_1(x) - f_2(x)\big)$$
$$= h\left(\tfrac{1}{2}\left(1+\tfrac{1}{\sqrt{3}}\right)\right) \quad x^*$$
$$= x(-1+3x-2x^2) \quad x = x^*$$
$$6x^2 = 6x-1$$
$$2x^2 = 2x - \tfrac{1}{3}$$
$$= \tfrac{1}{2}\left(1+\tfrac{1}{\sqrt{3}}\right)\cdot\left(x - \tfrac{2}{3}\right)$$
$$= \tfrac{1}{2}\left(1+\tfrac{1}{\sqrt{3}}\right)\left(-\tfrac{1}{3} + \tfrac{1}{\sqrt{3}}\right)\tfrac{1}{2}$$
$$= \tfrac{1}{4}\left(-\cancel{\tfrac{1}{3}} - \tfrac{1}{3\sqrt{3}} + \tfrac{1}{\sqrt{3}} + \cancel{\tfrac{1}{3}}\right) = \tfrac{1}{4}\cdot\tfrac{2}{3\sqrt{3}} = \underline{\underline{\tfrac{1}{6\sqrt{3}}}}$$



$\frac{1}{2}\left(1+\frac{1}{\sqrt{3}}\right)$

c) $x \geq 1 \Rightarrow g(x) = \frac{1}{2}\left(x + \frac{2}{x}\right) \geq 1$ since

$g' = \frac{1}{2}\left(1 - \frac{2}{x^2}\right) = 0$ for $x = \sqrt{2}$



$g(\sqrt{2}) = \frac{1}{2}\left(\sqrt{2} + \frac{2}{\sqrt{2}}\right) = \sqrt{2}$

$\Rightarrow g(x) \geq \sqrt{2} \geq 1$

Hence, $g: [1,\infty) \to [1,\infty)$ continuous map.

It is enough to show that g is a contraction to see that it has a fixed point

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by the fixed pt. thm. ($[1,\infty)$ has Euclidean norm, given by $|x|$):

$$x, x' \geq 1 \Rightarrow |g(x)-g(x')| = \left| \tfrac{1}{2}\left(x+\tfrac{2}{x}\right) - \tfrac{1}{2}\left(x'+\tfrac{2}{x'}\right) \right|$$

$$= \frac{1}{2} \cdot \left| x - x' + \frac{2}{x} - \frac{2}{x'} \right|$$

$$= \frac{1}{2} \left| x - x' + \frac{2x' - 2x}{xx'} \right|$$

$$= \frac{1}{2} \left| x - x' + \frac{2}{xx'} (x' - x) \right|$$

$$= \frac{1}{2} \left| 1 - \frac{2}{xx'} \right| \cdot |x - x'|$$

$$\leq \frac{1}{2} \left| 1 - \frac{2}{1 \cdot 1} \right| \cdot |x - x'| \leq \frac{1}{2} |x - x'|$$

$\square$

Hence g is a contraction.