Exam	Final exam in DRE 7017 Mathematics, Ph.D.
Date	June 1st, 2017 at 0900 - 1200

Question 1.

We consider the function $f(x, y, z) = xz - y^2$.

(A) Determine if f is convex or concave.
(B) Find the maximum and the minimum value in the Lagrange problem

$$\max / \min f(x, y, z) = xz - y^2 \quad \text{when } x^2 + y^2 + z^2 = 6$$

if they exist. Justify your answer.

Question 2.

We consider the system of linear differential equations given by

$$\begin{aligned} \dot{x} &= y - 2 \\ \dot{y} &= z - 3 \\ \dot{z} &= 8x - 2y - 5z - 5 \end{aligned}$$

(A) Find the steady state $(\overline{x}, \overline{y}, \overline{z})$.
(B) Rewrite the system in the form $\mathbf{w}' = A\mathbf{w}$. Show that $\lambda = 1$ is an eigenvalue for A, and use this to find all eigenvalues for A.
(C) Solve the system of linear differential equations.
(D) Describe all initial states (x_0, y_0, z_0) such that $(x, y, z) \to (\overline{x}, \overline{y}, \overline{z})$ when $t \to \infty$, and determine whether $(x_0, y_0, z_0) = (4, 1, 1)$ is among these initial states.

Question 3.

We consider the optimal control problem

$$\min \sum_{t=0}^{n} \left(1 + x_t^2 - u_t^2\right) \quad \text{when} \quad \begin{cases} x_0 = 2 \\ x_{t+1} = x_t(1 + u_t) \\ u_t \in U \end{cases}$$

with control region $U = [0, 1]$.

(A) Show that $x_t \geq 2$ for $1 \leq t \leq n$.
(B) Solve the optimal control problem when $n = 2$. What is the minimal value?

Question 4.

Let $V = C([0, 1])$ be the function space of continuous functions on the unit interval $[0, 1]$, and equip V with the sup norm

$$||f|| = \sup_{x \in [0,1]} |f(x)|$$

and the corresponding metric $d(f, g) = ||f - g||$ for all $f, g \in V$. We consider the functions $p(x) = e^x$ and $q(x) = 2 + x$ in V.

(A) Compute $||p||$, $||q||$, and $d(p, q)$.
(B) Describe the set of functions $\{f \in V : d(f, p) = 0\}$.