Differential equations - quick and dirty

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1. What is a differential equation?

A differential equation is an equation where a *function* is the unknown, and where one or more of its derivatives are involved.

Example 1. Find a function $y(x)$ such that $y' = 1$. This is a differential equation, and we know how to solve it: Integrating with respect to x on both sides of the equation gives

$$\int y'(x)dx = \int 1dx$$

which implies

$$y(x) + C_1 = x + C_2$$

where C_1 and C_2 are real numbers. Isolating $y(x)$ solves the equation:

$$y(x) = x + (C_2 - C_1) = x + C$$

where C is some real number.

Whenever we are asked to *prove* that a given function is a solution to a given differential equation, we insert the given function into the equation and see that it fits:

Example 2. Prove that $y(x) = e^{5x}$ is a solution to the differential equation

$$y' - 5y = 0$$

In our case, $y(x) = e^{5x}$, and therefore $y'(x) = 5e^{5x}$. Thus

$$y'(x) - 5y(x) = 5e^{5x} - 5e^{5x} = 0$$

Thus the given function $y(x) = e^{5x}$ fits into the given differential equation $y' - 5y = 0$, and is therefore a solution to the equation.

A function is a solution to a given differential equation of and only if it fits into the equation:

Example 3 (Example 2 continued)**.** Consider the function $y(x) = x^2$. Then $y'(x) = 2x$ and

$$y'(x) - 5y(x) = 2x - 5x^2 \neq 0$$

Thus $y(x)$ does not fit into the equation $y' - 5y = 0$, and is therefore not a solution to this equation.

2. Separable differential equations

A differential equation on the form

$$y'(x) = f(x)g(y) \tag{1}$$

is called *separable*.

Example 4.

(a) $y' = x \cdot y$. This is a separable differential equation with $f(x) = x$ and $g(y) = y$. We can *separate* the variable x from the function y: The given equation is equivalent to $\frac{y'}{y} = x$

(b) $y' = x \cdot y - x$. The right hand side of the equation equals $x(y-1)$, and this is therefore a separable differential equation with $f(x) = x$ and $g(y) = y - 1$. Again, we can *separate* the variable x from the function y: The given equation is equivalent to $\frac{y'}{y-1} = x$

So how do we solve separable differential equations? First, we try to solve the differential equation from example 4(a): We must find a function $y(x)$ such that $\frac{y'}{y} = x$. Recall that y' is short for $\frac{dy}{dx}$, and the equation is

$$\frac{\frac{dy}{dx}}{y} = x$$

and pretending that $\frac{dy}{dx}$ is a fraction, we can write this

$$\frac{dy}{y} = xdx$$

Integrating on both sides gives

$$\int \frac{dy}{y} = \int xdx$$

that is

$$\ln|y| + C_1 = \frac{1}{2}x^2 + C_2$$

Move C_1 over to the right hand side and let $C = C_2 - C_1$:

$$\ln|y| = \frac{1}{2}x^2 + C$$

and therefore

$$|y| = e^{\frac{1}{2}x^2 + C} = |K|e^{\frac{1}{2}x^2}$$

where $|K| = e^C$. This implies that

$$y = Ke^{\frac{1}{2}x^2}$$

where K is a real number. We get one solution for every choice of the real number K. If we use the symbols $\frac{dy}{dx}$ for y', the general method for solving separable differential equations looks like this:

General method for solving separable differential equations:

Step 1 You have an equation of the form

$$\frac{dy}{dx} = f(x)g(y)$$

Step 2 Separate the variable x from the function y:

$$\frac{dy}{g(y)} = f(x)dx$$

Step 3 Integrate:

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

Calculate the integrals. You get an equation involving x and y. If possible, isolate y, i.e. express y in terms of x. You get infinitely many solutions, one for each choice of the integration constant C.

Step 4 In addition to the solutions from Step 3, there may be constant solutions. If there exist numbers a such that $g(a) = 0$, then $y(x) = a$ is a solution for every such a.

Example 5. Solve the differential equation

$$y^2 \frac{dy}{dx} = x + 1$$

Then find the solution that passes through the point $(x, y) = (1, 1)$. First of all, this is a separable differential equation, and Step 2 from above becomes

$$y^2 dy = (x + 1)dx$$

We integrate:

$$\int y^2 dy = \int (x + 1)dx$$

Calculating the integrals gives

$$\frac{1}{3}y^3 = \frac{1}{2}x^2 + x + C$$

Notice that there should have been a constant of integration on the left hand side, too, but we include this is the C on the right. It is possible to isolate y from this equation:

$$y = y(x) = \sqrt[3]{3\left(\frac{1}{2}x^2 + x + C\right)} = \left[3\left(\frac{1}{2}x^2 + x + C\right)\right]^{\frac{1}{3}}$$

Every choice of C gives a different solution to the equation. We want to find the unique solution which has a graph passing through the point $(1, 1)$. Thus $y(1)$ must be 1, which implies the equation

$$\sqrt[3]{3\left(\frac{1}{2} + 1 + C\right)} = 1$$

A cleaned up version of this is

$$\sqrt[3]{\frac{9}{2} + 3C} = 1$$

Third power on both sides:

$$\frac{9}{2} + 3C = 1$$

Solving for C gives

$$C = \frac{1 - \frac{9}{2}}{3} = -\frac{7}{6}$$

The special solution passing though $(1, 1)$ is

$$y(x) = \sqrt[3]{3\left(\frac{1}{2}x^2 + x - \frac{7}{6}\right)}$$

3. First order linear differential equations

A differential equation of the form

$$y' + a(x) \cdot y = b(x) \tag{2}$$

where a and b are continuous functions of x, are called *first order linear differential equations*. There are three separate cases of such differential equations:

3.1. **Case 1: a and b are constants.** In this case, equation 2 becomes

$$y' + ay = b \qquad \text{where } a \text{ and } b \text{ are constants, } a \neq 0$$

We will now perform a trick which nobody expects you to understand how we came up with. Hopefully, you will see the beauty, though: If we multiply both sides of this equation by e^{ax}, we get the equation

$$y'e^{ax} + aye^{ax} = be^{ax} \tag{3}$$

The factor e^{ax} is called *the integrating factor*. Now, we recognize the right hand side of equation 3 as the derivative of the product ye^{ax}:

$$\frac{d}{dx}(ye^{ax}) = y'e^{ax} + y(e^{ax})' = y'e^{ax} + aye^{ax}$$

Thus equation 3 is equivalent to the equation

$$\frac{d}{dx}(ye^{ax}) = be^{ax}$$

and integrating both sides gives

$$ye^{ax} = \int be^{ax} dx = \frac{b}{a}e^{ax} + C$$

where C is any constant. Now, we can isolate y on the left hand side:

$$y = y(x) = \frac{\frac{b}{a}e^{ax} + C}{e^{ax}} = \frac{b}{a} + Ce^{-ax}$$

Notice that there is one solution for every choice of the constant C. To summarize:

$$\boxed{y' + ay = b \Leftrightarrow y = y(x) = Ce^{-ax} + \tfrac{b}{a} \text{ for any constant } C}$$

Example 6. Solve the differential equation

$$y' + 3y = 5$$

Here, $a = 3$ and $b = 5$ and the formula above implies

$$y(x) = Ce^{-3x} + \frac{3}{5}$$

for any constant C.

3.2. **Case 2: a is constant, b is a function.** In this case, equation 2 becomes

$$y' + ay = b(x)$$

The method of multiplying both sides with the integrating factor e^{ax} works in this case, too:

$$y'e^{ax} + ay^{ax} = b(x)e^{ax} \tag{4}$$

The left hand side is the derivative of the product ye^{ax}, and equation 4 is therefore equivalent to

$$\frac{d}{dx}(ye^{ax}) = b(x)e^{ax}$$

Integration gives

$$ye^{ax} = \int b(x)e^{ax}dx + C$$

and we can isolate y:

$$y = y(x) = Ce^{-ax} + e^{-ax}\int b(x)e^{ax}dx$$

When calculating the integral, you do not need to include an integration constant, as this is already included in C. To summarize:

$$y' + ay = b(x) \Leftrightarrow y = y(x) = Ce^{-ax} + e^{-ax}\int e^{ax}b(x)dx$$

Example 7. Solve the differential equation $y' + 2y = x$. In this case, $a = 2$ and $b = b(x) = x$. The formula implies

$$y = y(x) = Ce^{-2x} + e^{-2x}\int e^{2x}xdx \tag{5}$$

We calculate the integral using the technique of integration by parts: Let $u = x$ and $v' = e^{2x}$. Then $u' = 1$ and $v = \frac{1}{2}e^{2x}$. Integration by parts gives

$$\int \underbrace{e^{2x}}_{v'}\underbrace{x}_{u}dx = \underbrace{\frac{1}{2}xe^{2x}}_{u\cdot v} - \int \underbrace{\frac{1}{2}e^{2x}}_{v}\cdot\underbrace{1}_{u'}dx$$

that is

$$\int e^{2x}xdx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}$$

Inserting this into equation 5 gives

$$\begin{aligned} y = y(x) &= Ce^{-2x} + e^{-2x}\left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right) \\ &= Ce^{-2x} + \frac{1}{2}x - \frac{1}{4} \end{aligned}$$

3.3. **Case 3: Both a and b are functions.** We will not include the explanation for the solution in this case. We just choose to believe the following:

$$y' + a(x)y = b(x) \Leftrightarrow y = y(x) = e^{-\int a(x)dx}\left(C + \int e^{\int a(x)dx} b(x)dx\right)$$

Example 8. Solve the differential equation $y' - \frac{1}{x}y = x$ where $x > 0$. Now $a(x) = -\frac{1}{x}$ and $b(x) = x$. Thus

$$\int a(x)dx = \int -\frac{1}{x}dx = -\ln(x)$$

Since $e^{-\ln(x)} = \frac{1}{x}$, and $e^{\ln(x)} = x$, the formula implies

$$\begin{aligned} y = y(x) &= x\left(C + \int \frac{1}{x} \cdot x dx\right) \\ &= x\left(C + x\right) \\ &= Cx + x^2 \end{aligned}$$

4. Second order linear differential equations

A second order linear differential equation is of the form

$$a(x)y'' + b(x)y' + c(x)y = d(x) \tag{6}$$

where a, b, c and d are continuous functions of x. We will only give a method for solving such equations in the case when a, b, c and d are constants. In the case when $d = 0$, we are facing a *homogeneous second order linear differential equation with constant coefficients*:

$$ay'' + by' + cy = 0$$

We will not explain, but simply believe the following method for solving such differential equations:

Method for solving homogeneous second order linear differential equations with constant coefficients:

Step 1 You start with an equation of the type

$$ay'' + by' + cy = 0$$

Step 2 Write down *the characteristic equation*

$$ar^2 + br + c = 0$$

and solve for r.

Step 3 There are three cases:

(1) If $b^2 - 4ac > 0$, the characteristic equation has two different solutions r_1 and r_2. Then

$$y(x) = Ae^{r_1 x} + Be^{r_2 x}$$

for constants A and B.

(2) If $b^2 - 4ac = 0$, the characteristic equation as one solution r. Then

$$y(x) = (A + Bx)e^{rx}$$

for constants A and B.

(3) If $a^2 - 4ac < 0$, a solution exists, but it is not part of this course.

What if $d(x)$ in equation 6 is a nonzero constant? In this case, the differential equation looks like this:

$$ay'' + by' + cy = d \tag{7}$$

Observe that the constant function $\overline{y}(x) = \frac{d}{c}$ is a solution:

$$\overline{y}'' = 0 \qquad \text{and} \qquad \overline{y}' = 0$$

and therefore

$$a\overline{y}'' + b\overline{y}' + c\overline{y} = 0 + 0 + c\overline{y} = d$$

Thus $\overline{y}(x) = \frac{d}{c}$ fits into the equation, and is therefore a solution. To find the general solution to the differential equation in 7, follow the following procedure:

Method for solving nonhomogeneous second order linear differential equations with constant coefficients:

Step 1 You start with an equation of the type

$$ay'' + by' + cy = d$$

Step 2 Solve the corresponding homogeneous equation using the method above. Call this solution $y_h(x)$.

Step 3 You know that the constant function $\overline{y}(x) = \frac{d}{c}$ is a solution

Step 4 The general solution is

$$y(x) = y_h(x) + \overline{y}(x)$$

5. A note on notation

If the variable is time t, we often denote the function by x instead of y. Furthermore, if $x(t)$ is a twice differentiable function of t, we denote the first derivative of x with respect to t by $\dot{x}$, the second derivative with respect to t by $\ddot{x}$ and so on. Thus when $x(t)$ is the function and t is the variable,

$$\dot{x} = \frac{dx}{dt} \qquad \text{and} \qquad \ddot{x} = \frac{d^2x}{dt^2}$$

and the differential equation

$$a\ddot{x} + b\dot{x} + cx = 0$$

is the same differential equation as

$$ax'' + bx' + cx = 0$$

6. Want to know more?

If you want to know more about differential equations, you can read chapters 1, 2 and 3 in *Matematisk Analyse, bind 2* (Sydsæter, Seierstad, Strøm).