FORELESNING 8 ELE 3719

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PLAN:

1. Oppgaver: Oppgaveark 4
2. Repetisjon: – Viktige formler (forventning / varians) [R] 2.4
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3. Viktige resultat [R] 2.8
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Oppgaveark 4:

48) $X \geq 0$

$g(t)$ funksjon s.a. $g(0)=0$

Vis at:

$$E[g(X)] = \int_0^\infty P(X>t)\cdot g'(t)\,dt$$

$$\Updownarrow$$

$$\int_0^\infty g(t) f(t)\,dt = \int_0^\infty P(X>t)\, g'(t)\,dt$$

Bevis:

$$1-P(X\leq t) = 1-F(t)$$

$$\int_0^\infty \underbrace{P(X>t)}_{u}\cdot \underbrace{g'(t)}_{v'}\,dt \underset{\text{delvis}}{=} \Big[u\cdot v\Big]_0^\infty - \int_0^\infty u' v\,dt$$

$$= \Big[P(X>t)\cdot g(t)\Big]_0^\infty - \int_0^\infty \frac{d}{dt}\big(1-F(t)\big)\cdot g(t)\,dt$$

$$= \Big[P(X>t)\cdot g(t)\Big]_0^\infty + \int_0^\infty f(t)\,g(t)\,dt$$

$$= P(X>\infty)\cdot g(\infty) - P(X>0)\cdot g(0) + E(g(X))$$

$$= \lim_{t\to\infty} \underset{\substack{\downarrow\\0}}{P(X>t)}\cdot \underset{\substack{\downarrow\\?}}{g(t)} - 1\cdot 0 + E(g(X))$$

$$= E(g(X))$$

Man kan vise (men vanskelig) at hvis $E(g(X))$ er endelig så er

$$\lim_{t\to\infty} P(X>t)\cdot g(t) = 0$$

52.)

a) $M = \max$ $\qquad F_M(x) = x^n$

$f_M(x) = n \cdot x^{n-1} \quad , 0 \leq x \leq 1$

$$E[M] = \int_{-\infty}^{\infty} \cancel{x} \cdot f_m(x)\, dx$$

$$= \int_0^1 x \cdot n x^{n-1}\, dx = \int_0^1 n \cdot x^n\, dx$$

$$= n \cdot \left[\frac{1}{n+1} x^{n+1}\right]_0^1 = n \cdot \left(\frac{1}{n+1} \cdot 1^{n+1} - \frac{1}{n+1} \cdot 0^{n+1}\right)$$

$$= \frac{n}{n+1}$$

② Repetisjon: Viktige formler

$E(aX+b) = a\,E(X)+b$

$E(X^2) \neq E(X)^2$

$\mu = E(X) = \bar{X}$

$Var(X) = E(X^2) - E(X)^2$

$Var(aX+b) = a^2\,Var(X)$

$Var(X) \geq 0$

$\sigma^2 = Var(X)$

$\sigma = \sqrt{Var(X)}$

σ = std. avvik

Diskret		Kontinuerlig	
binomisk (n,p)	$E(X) = np$, $Var(X) = np(1-p)$	uniform (a,b)	$E(X) = \frac{a+b}{2}$, $Var(X) = \frac{(b-a)^2}{12}$
geometrisk (p)	$E(X) = 1/p$, $Var(X) = \frac{1-p}{p^2}$	eksponensial $(\lambda > 0)$	$E(X) = 1/\lambda$, $Var(X) = 1/\lambda^2$
Poisson (λ)	$E(X) = \lambda$, $Var(X) = \lambda$	normal (μ, σ^2)	$E(X) = \mu$, $Var(X) = \sigma^2$

③ Teoremer [R] 2.8

i) Markov's ulikhet: $p(X \geq a) \leq \frac{E(X)}{a}$

$X \geq 0$, $a > 0$

Bevis:

$$E(X) = \int_0^\infty x f(x)\, dx = \int_0^a x f(x)\, dx + \int_a^\infty x f(x)\, dx$$

$$\geq \int_a^\infty x f(x)\, dx \geq \int_a^\infty a f(x)\, dx = a \cdot \int_a^\infty f(x)\, dx$$

$$= a \cdot p(X \geq a)$$

Dvs:

$$E(X) \geq a \cdot p(X \geq a) \Rightarrow \frac{E(X)}{a} \geq p(X \geq a)$$

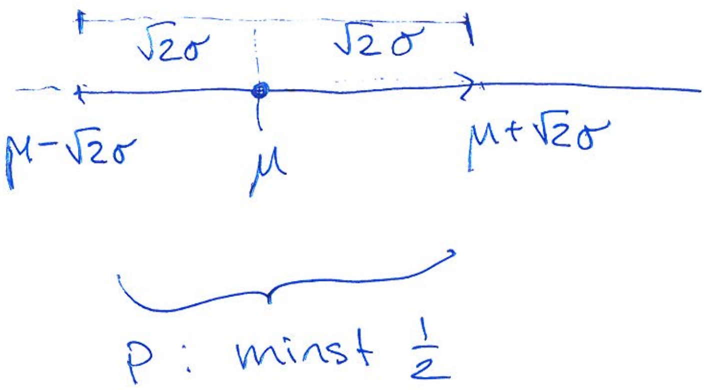
ii) Chebyshev's ulikhet: $(k > 0)$

$$p(|X - \mu| \geq k \cdot \sigma) \leq \frac{1}{k^2}$$

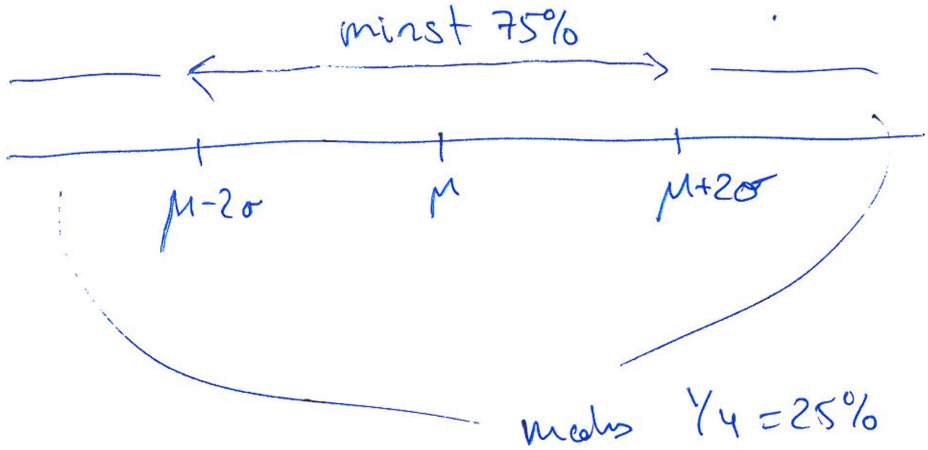
C's ulikhet følger fra M's ulikhet.

(utledning: [R] 2.8)

$\underline{k = \sqrt{2}}$: $\quad p(|X-\mu| \geq \sqrt{2}\sigma) \leq \frac{1}{(\sqrt{2})^2} = \frac{1}{2}$



$\underline{k = 2}$: $\quad p(|X-\mu| \geq 2\sigma) \leq \frac{1}{4}$



④ Simultant fordelte stokastiske variable

— Diskret tilfelle

Eks: X og Y er diskrete simultant fordelte stokastiske variable

$$f(x,y) = P(X=x, Y=y) = P(X=x \text{ og } Y=y)$$

(Sannsynlighetstettheten)

	Y=1	Y=2	Y=3
X=1	0.20	0.15	0.20
X=2	0.10	0.15	0.20

← Sannsynlis-hetene

Dvs:

$f(1,1) = 0.20 \quad f(1,2) = 0.15 \quad f(1,3) = 0.20$

$f(2,1) = 0.10 \quad f(2,2) = 0.15 \quad f(2,3) = 0.20$

$$p(X=2) = f(2,1) + f(2,2) + f(2,3)$$
$$= 0.10 + 0.15 + 0.20 = 0.45$$

$$E(X) = 1 \cdot p(X=1) + 2 \cdot p(X=2)$$
$$= 1 \cdot \underline{0.55} + 2 \cdot \underline{0.45} = \underline{\underline{1.45}}$$

Alt: $E(X) = 1 \cdot 0.20 + 1 \cdot 0.15 + 1 \cdot 0.20$
$+ 2 \cdot 0.10 + 2 \cdot 0.15 + 2 \cdot 0.20 = \underline{\underline{1.45}}$

Defn: Sannsynlighetstetthet $f(x,y)$

$$f_X(x) = \sum_y f(x,y) \qquad f_Y(y) = \sum_x f(x,y)$$

sann. tetthet for X sann. tetthet for Y

$$E(X) = \sum_x x \cdot f_X(x) = \sum_{x,y} x \cdot f(x,y)$$

$$E(Y) = \sum_y y \cdot f_Y(y) = \sum_{x,y} y \cdot f(x,y)$$

$$E[g(X,Y)] = \sum_{x,y} g(x,y) \cdot f(x,y)$$

Eks:

	Y=1	2	3
X=1	0.20	0.15	0.20
2	0.10	0.15	0.20

$$E(XY) = \sum_{x,y} xy \cdot f(x,y)$$

$$= 1\cdot 1\cdot 0.20 + 1\cdot 2\cdot 0.15 + 1\cdot 3\cdot 0.20$$
$$+ 2\cdot 1\cdot 0.10 + 2\cdot 2\cdot 0.15 + 2\cdot 3\cdot 0.20$$

$$= 0.20 + 0.30 + 0.60 + 0.20 + 0.60 + 1.20 = \underline{\underline{3.10}}$$

Defn: X og Y simultant fordelt

$$\left.\begin{array}{l} Var(X) = E(X^2) - E(X)^2 \\ Var(Y) = E(Y^2) - E(Y)^2 \end{array}\right\} \text{ varians}$$

$$Cov(X,Y) := E[(X-\mu_X)(Y-\mu_Y)] = \underline{E[XY] - E[X]\cdot E[Y]}$$

Krav til en simultan sannsynlighetstetthet:

i) $f(x,y) \geq 0$ for alle x,y

ii) $\sum_{x,y} f(x,y) = 1$

Eks:

$X \backslash Y$	1	2	3
1	0.2	0.15	0.20
2	0.1	0.15	0.2

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X) \cdot E(Y) \\ &= 3.1 - 1.45 \cdot 2.1 \\ &= 3.1 - 3.045 = \underline{0.055} \end{aligned}$$

$\underline{E(Y) = ?}$ $\quad f_Y(1) = 0.3 \quad f_Y(2) = 0.3 \quad f_Y(3) = 0.4$

$$\begin{aligned} E(Y) &= 1 \cdot 0.3 + 2 \cdot 0.3 + 3 \cdot 0.4 \\ &= 0.3 + 0.6 + 1.2 = \underline{2.1} \end{aligned}$$