Oppgave 1

Avgjør i hvert tilfelle om funksjonen er konveks, konkav, begge deler eller ingen av delene:

a) $f(x, y) = x + y$

b) $f(x, y) = 3x - y$

c) $f(x, y) = e^x + e^y$

d) $f(x, y) = e^{-x-y}$

e) $f(x, y) = \ln(x) + \ln(y)$

a) $f'_x = 1$ $\quad f''_{xx} = f''_{xy} = f''_{yy} = 0 \Rightarrow$ Både konveks og konkav
$f'_y = 1$

b) $f'_x = 3$ $\quad f''_{xx} = f''_{xy} = f''_{yy} = 0 \Rightarrow$ Både konveks og konkav
$f'_y = -1$

c) $f'_x = e^x$ $\quad f''_{xx} = e^x \quad f''_{yy} = e^y$
$f'_y = e^y$ $\quad f''_{xy} = 0$
$\Rightarrow e^x \cdot e^y - 0^2 > 0 \Rightarrow$ konveks
$e^x, e^y > 0$

d) $f'_x = -e^{-x-y}$ $\quad f''_{xx} = f''_{xy} = f''_{yy} = e^{-x-y}$
$f'_y = -e^{-x-y}$
$\Rightarrow f''_{xx} \cdot f''_{yy} - (f''_{xy})^2 = 0 \Rightarrow$ konveks
$f''_{xx}, f''_{yy} > 0$

e) $f'_x = \frac{1}{x}$ $\quad f''_{xx} = -1/x^2$ $\quad f''_{xy} = 0$
$f'_y = \frac{1}{y}$ $\quad f''_{yy} = -1/y^2$
$\Rightarrow f''_{xx} \cdot f''_{yy} - f''^{\,2}_{xy} = \frac{1}{x^2 y^2} > 0 \Rightarrow$ konkav
$f''_{xx}, f''_{yy} < 0$

OPPGAVE 2

Vis at funksjonen $f(x, y) = x^2 + 4xy + 4y^2 + e^y - y$ er konveks.

$f'_x = 2x + 4y$

$f'_y = 4x + e^y - 1 + 8y$

$f''_{xx} = 2$

$f''_{yy} = 8 + e^y$

$f''_{xy} = 4$

$f''_{xx} f''_{yy} - f''^{\,2}_{xy}$

$= 2 \cdot (8 + e^y) - 4^2$

$= 2e^y > 0$

$f''_{xx},\ f''_{yy} > 0$

$\Downarrow$

<u>konveks</u>

Oppgave 3

Vi betrakter variasjonsproblemet

$$\min \int_0^1 (t\dot{y} + \dot{y}^2)\,\mathrm{d}t, \quad y(0) = 1,\ y(1) = 0$$

a) Finn Euler-likningen og løs den. Vis at løsningen gir et minimum.

b) Finn den løsningen som tilfredsstiller initialbetingelsene.

a) $F = t\dot{y} + \dot{y}^2$

$$\frac{\partial F}{\partial y} = 0 \qquad \frac{\partial F}{\partial \dot{y}} = t + 2\dot{y} \qquad \frac{d}{dt}\frac{\partial F}{\partial \dot{y}} = 1 + 2\ddot{y}$$

$$0 - (1 + 2\ddot{y}) = 0 \Leftrightarrow 2\ddot{y} + 1 = 0 \Leftrightarrow \ddot{y} = -\frac{1}{2}$$

<u>Løsn:</u> $\dot{y} = \int -\frac{1}{2}\,dt = -\frac{1}{2}t + C_1$

$$y = -\frac{1}{4}t^2 + C_1 t + C_2$$

$$\left.\begin{array}{l} F''_{yy} = 0 \\ F''_{y\dot{y}} = 0 \\ F''_{\dot{y}\dot{y}} = 2 \end{array}\right\} \Rightarrow F \text{ konveks i } (y, \dot{y}) \Rightarrow \text{Løsn gir } \underline{\text{min}}.$$

b) $y(0) = 1$: $\quad \underline{C_2 = 1}$

$y(1) = 0$: $\quad -\frac{1}{4} + C_1 + 1 = 0 \Rightarrow \underline{C_1 = -3/4}$

$$\underline{\underline{y = -\frac{1}{4}t^2 - \frac{3}{4}t + 1}}$$

OPPGAVE 4

Finn Euler-likningen som er tilordnet integralet

$$\min \int_{t_0}^{t_1} F(t, y, \dot{y})\, dt$$

i hvert tilfelle:

a) $F(t, y, \dot{y}) = 2ty + 3y\dot{y} + t\dot{y}^2$

b) $F(t, y, \dot{y}) = -e^{\dot{y}-ay}$

c) $F(t, y, \dot{y}) = ((y - \dot{y})^2 + y^2)e^{-at}$

$\cancel{a)\ \frac{\partial F}{\partial y} = 2t + 3\dot{y} \quad \frac{\partial F}{\partial \dot{y}} = -e^{\dot{y}-ay}\cdot(-a) \quad \frac{d}{dt}\frac{\partial F}{\partial \dot{y}} = a\cdot e^{\dot{y}-ay}\cdot(\ddot{y}-a\dot{y}) \quad 2t+3\dot{y} - a(\ddot{y}-a\dot{y})e^{\dot{y}-ay}}$

a)

$$F'_y = 2t + 3\dot{y}$$
$$F'_{\dot{y}} = 3y + 2t\dot{y}$$
$$\frac{d}{dt}F'_{\dot{y}} = 3\dot{y} + 2t\cdot\ddot{y} + 2\dot{y}$$

$$2t + 3\dot{y} - (5\dot{y} + 2t\ddot{y}) = 0$$
$$\underline{2t - 2\dot{y} - 2t\cdot\ddot{y} = 0}$$

b)

$$F'_y = -e^{\dot{y}-ay}\cdot(-a) = ae^{\dot{y}-ay}$$
$$F'_{\dot{y}} = -e^{\dot{y}-ay}\cdot 1 = -e^{\dot{y}-ay}$$
$$\frac{d}{dt}F'_{\dot{y}} = -e^{\dot{y}-ay}\cdot(\ddot{y} - a\dot{y})$$

$$ae^{\dot{y}-ay} + e^{\dot{y}-ay}(\ddot{y} - a\dot{y}) = 0$$
$$\underline{e^{\dot{y}-ay}\cdot(\ddot{y} - a\dot{y} + a) = 0}$$

c)

$$F'_y = (2(y-\dot{y}) + 2y)e^{-at}$$
$$F'_{\dot{y}} = -2(y-\dot{y})e^{-at}$$
$$\frac{d}{dt}F'_{\dot{y}} = -2(\dot{y}-\ddot{y})e^{-at} - 2(y-\dot{y})e^{-at}\cdot(-a) = e^{-at}(2\ddot{y} - (2+2a)\dot{y} + 2ay)$$

$$(4y - 2\dot{y})e^{-at} - (2\ddot{y} - (2+2a)\dot{y} + 2ay)e^{-at} = 0$$
$$\underline{(-2\ddot{y} + 2a\dot{y} + (4-2a)y)e^{-at} = 0}$$

OPPGAVE 5

Vis at Euler-likningen som svarer til variasjonsproblemet

$$\min \int_a^b (x^2 + tx\dot{x} + t^2\dot{x}^2)\,dt$$

er gitt ved $t^2\ddot{x} + 2t\dot{x} - \frac{1}{2}x = 0$.

$$F(t, x, \dot{x}) = x^2 + tx\dot{x} + t^2\dot{x}^2$$

$$F'_x = 2x + t\dot{x}$$

$$F'_{\dot{x}} = tx + 2t^2\dot{x}$$

$$\frac{d}{dt}F'_{\dot{x}} = 1\cdot x + t\cdot\dot{x} + 4t\cdot\dot{x} + 2t^2\ddot{x}$$

$$= x + 5t\cdot\dot{x} + 2t\ddot{x}$$

Euler:

$$(2x + t\dot{x}) - (x + 5t\dot{x} + 2t^2\ddot{x}) = 0$$

$$+x - 4t\dot{x} - 2t^2\ddot{x} = 0$$

$$t^2\ddot{x} + 2t\dot{x} \div \frac{1}{2}x = 0$$

OPPGAVE 6

a) Løs differensiallikningen $\ddot{y} + \frac{1}{t}\dot{y} = 1$.

b) Finn Euler-likninen som svarer til variasjonsproblemet

$$\min \int_1^2 (2ty + 3y\dot{y} + t\dot{y}^2)\,dt, \quad y(1) = 0,\ y(2) = 1$$

og bestem løsningen som tilfredsstiller initialbetingelsene.

a) $\ddot{y} + \frac{1}{t}\dot{y} = 1 \qquad (u = \dot{y})$

$\dot{u} + \frac{1}{t}u = 1 \qquad IF = e^{\int 1/t\,dt} = e^{\ln t} = t$

$(tu)' = t$

$tu = \int t\,dt = \frac{1}{2}t^2 + C_1$

$u = \frac{1}{2}t + \frac{C_1}{t} \qquad \dot{y} = \frac{1}{2}t + \frac{C_1}{t}$

$\underline{y = \frac{1}{4}t^2 + C_1 \ln t + C_2}$

b) $F = 2ty + 3y\dot{y} + t\dot{y}^2$

$F'_y = 2t + 3\dot{y}$

$F'_{\dot{y}} = 3y + 2t\dot{y}$

$\frac{d}{dt}F'_{\dot{y}} = 3\dot{y} + 2\dot{y} + 2t\ddot{y}$

$(2t + 3\dot{y}) - (3\dot{y} + 2\dot{y} + 2t\ddot{y}) = 0$

$2t - 2\dot{y} - 2t\ddot{y} = 0$

$\underline{\underline{\ddot{y} + \frac{1}{t}\dot{y} = 1}}$

$\underline{\underline{y = \frac{1}{4}t^2 + C_1 \ln t + C_2}}$

$y(1) = 0: \quad 0 = \frac{1}{4}\cdot 1^2 + C_1 \ln 1 + C_2 \Rightarrow C_2 = -1/4$

$y(2) = 1: \quad 1 = \frac{1}{4}\cdot 2^2 + C_1 \cdot \ln 2 - 1/4 \Rightarrow C_1 \ln 2 = 1/4$

$C_1 = \frac{1}{4\cdot \ln 2}$

$$\underline{\underline{y = \frac{1}{4}t^2 + \frac{1}{4\cdot \ln 2}\cdot \ln t - \frac{1}{4}}}$$

Oppgave 7

Vi betrakter variasjonsproblemet

$$\max \int_0^T e^{-t/4} \ln(2K - \dot{K})\,dt, \quad K(0) = K_0,\ K(T) = K_T$$

a) Vis at funksjonen $F(t, K, \dot{K}) = e^{-t/4} \ln(2K - \dot{K})$ er konkav som funksjon i $(K, \dot{K})$.

b) Vis at Euler-likningen kan skrives på formen $a\ddot{K} + b\dot{K} + cK = 0$ der a, b, c er konstanter.

c) Løs variasjonsproblemet.

a) $F = e^{-t/4} (\ln(2K-\dot K)$

$F'_K = 2e^{-t/4} \cdot \frac{1}{2K-\dot K}$

$F'_{\dot K} = -e^{-t/4} \cdot \frac{1}{2K-\dot K}$

$F''_{KK} = e^{-t/4}(2K-\dot K)^{-2} \cdot (-4)$

$F''_{K\dot K} = e^{-t/4}(2K-\dot K)^{-2} \cdot (2)$

$F''_{\dot K\dot K} = e^{-t/4}(2K-\dot K)^{-2} \cdot (-1)$

$F''_{KK}, F''_{\dot K\dot K} < 0$

$F''_{KK} \cdot F''_{\dot K\dot K} - (F''_{K\dot K})^2 = \left[e^{-t/4}(2K-\dot K)^{-2}\right]^2 ((-4)(-1)-2^2)$

$= 0 \geq 0$

$\Rightarrow F$ konkav i $(y, \dot y)$

b) $\frac{d}{dt} F'_{\dot K} = \frac{1}{4} e^{-t/4} \cdot \frac{1}{2K-\dot K}$

$- e^{-t/4} \cdot (2K-\dot K)^{-2} \cdot (-1) \cdot (2\dot K - \ddot K)$

Euler:

$2e^{-t/4}(2K-\dot K)^{-1} = \frac{1}{4} e^{-t/4} (2K-\dot K)^{-1} + (2\dot K - \ddot K) e^{-t/4}(2K-\dot K)^{-2} \quad \Big| \cdot \frac{(2K-\dot K)^2}{e^{-t/4}}$

$2(2K-\dot K) = \frac{1}{4}\cdot(2K-\dot K) + (2\dot K - \ddot K)$

$\ddot K - \frac{15}{4}\dot K + \frac{7}{2}K = 0$

c) $\ddot K - \frac{15}{4}\dot K + \frac{7}{2}K = 0$

$r^2 - \frac{15}{4}r + \frac{7}{2} = 0$

$r = \frac{15/4 \pm \sqrt{(15/4)^2 - 4\cdot(7/2)}}{2}$

$r = \frac{15}{8} \pm \frac{1}{8}\sqrt{15^2 - 28\cdot 8}$

$= \frac{15}{8} \pm \frac{1}{8}\cdot\sqrt{1} = \frac{16}{8}, \frac{14}{8} = 2, 7/4$

$K = C_1 \cdot e^{2t} + C_2 e^{7/4 t}$

$K(0) = K_0$:

$K_0 = C_1 e^0 + C_2 e^0$

$K_0 = C_1 + C_2$

$K(T) = K_T$:

$K_T = C_1 e^{2T} + C_2 e^{7T/4}$

$\Downarrow$

$C_2 = K_0 - C_1$

$K_T = C_1 e^{2T} + (K_0 - C_1) e^{\frac{7T}{4}}$

$K_T - K_0 e^{\frac{7}{4}T} = C_1 (e^{2T} - e^{7/4 T})$

$C_1 = \frac{K_T - K_0 e^{7/4 T}}{e^{2T} - e^{7/4 T}}$

$C_2 = K_0 - C_1 = \frac{K_0 e^{2T} - K_T}{e^{2T} - e^{7/4 T}}$

$$K = \frac{K_T - K_0 e^{\frac{7}{4}T}}{e^{2T} - e^{\frac{7}{4}T}} e^{2t} + \frac{K_0 e^{2t} - K_T}{e^{2T} - e^{\frac{7}{4}T}} e^{\frac{7}{4}t}$$