Oppgaveark 9
ELE 3719 Matematikk Valgfag

Handelshøyskolen BI

Oppgaver

1. Finn den generelle løsningen av $y' + \frac{1}{2}y = \frac{1}{4}$. Er løsningen stabil? Bestem likevektstilstanden. Tegn noen typiske løsninger i et koordinatsystem.

2. Finn den generelle løsningen i hvert tilfelle:

a) $y' + y = 10$
b) $y' - 3y = 27$
c) $4y' + 5y = 100$

3. Finn i hvert enkelt tilfelle den generelle løsningen. Finn også den partikulære løsningen som tilfredsstiller $y(0) = 1$.

a) $y' - 3y = 5$
b) $3y' + 2y + 16 = 0$
c) $y' + 2y = t^2$

4. Finn den generelle løsningen:

a) $ty' + 2y + t = 0, \quad t \neq 0$
b) $y' - \frac{1}{4}y = t, \quad t > 0$
c) $y' - \frac{t}{t^2-1}y = t, \quad t > 1$

5. Finn i hvert enkelt tilfelle den generelle løsningen. Finn også den partikulære løsningen som tilfredsstiller den gitte initialbetingelsen.

a) $y' = 4(y-1)(y-3), \quad y(0) = 2$
b) $e^{2t}y' - y^2 - 2y = 1, \quad y(1) = 1$
c) $y' - \frac{t}{t^2-1}y = 0, \quad y(0) = 1$

6. Finn den generelle løsningen i hvert tilfelle:

a) $y'' = t$
b) $y'' = e^t + t^2$

7. Løs initialverdiproblemet $y'' = t^2 - t, \quad y(0) = 1,\ y'(0) = 2$.

8. Løs initialverdiproblemet $y'' = y' + t, \quad y(0) = 1,\ y'(1) = 2$.

9. Finn i hvert enkelt tilfelle den generelle løsningen:

a) $y'' - 3y = 0$
b) $y'' + 4y' + 8y = 0$
c) $3y'' + 8y' = 0$
d) $4y'' + 4y' + y = 0$
e) $y'' + y' - 6y = 0$

10. Løs differensiallikningen $y'' + y' - 6y = 7$.

11. Finn løsningen av differensiallikningen

$$y'' - 10y' + 25y = 4$$

som tilfredsstiller $y(0) = 29/25$ og $y(1) = 2e^5 + 4/25$.

12. Ta utgangspunkt i differensiallikningen $y'' + ay' + by = 0$, og anta at $a^2 - 4b = 0$ slik at den karakteristiske likningen har en dobbelrot r. La $y(t) = u(t)e^{rt}$, og vis at $y(t)$ er en løsning av differensiallikningen hvis og bare hvis $u'' = 0$. Konkluder fra dette at $y(t) = (A + Bt)e^{rt}$ er den generelle løsningen.

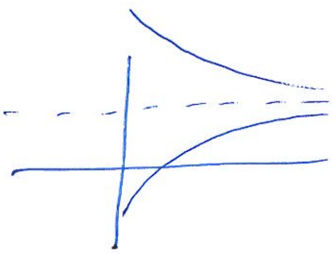
Løsning: Oppgaveark 9

1) $y' + \frac{1}{2}y = \frac{1}{4}$

$(ye^{\frac{1}{2}t})' = \frac{1}{4}e^{\frac{1}{2}t}$

$y = e^{-\frac{1}{2}t}\left(\frac{1}{2}e^{\frac{1}{2}t} + C\right)$

$= \underline{\frac{1}{2} + Ce^{-\frac{1}{2}t}}$



Stabil, $\lim_{t\to\infty} y(t) = \underline{1/2}$ for alle C

2) a) $y' + y = 10 \Rightarrow y = \underline{10 + C\cdot e^{-t}}$

b) $y' - 3y = 27 \quad y = \underline{-9 + Ce^{3t}}$

c) $4y' + 5y = 100 \quad y = \underline{20 + C\cdot e^{-5/4t}}$

3) a) $y' - 3y = 5 \Rightarrow y = -5/3 + Ce^{3t}$

$y(0) = 1$: $\quad 1 = -5/3 + C \Rightarrow C = 8/3 \Rightarrow y = \underline{-5/3 + 8/3\, e^{3t}}$

b) $3y' + 2y + 16 = 0 \Rightarrow y = -8 + Ce^{-2/3t}$

$y(0) = 1$: $\quad 1 = -8 + C \Rightarrow C = 9 \Rightarrow y = \underline{-8 + 9e^{-2/3t}}$

c) $y' + 2y = t^2 \Rightarrow y = e^{-2t}\int t^2 e^{2t}\,dt$

$= e^{-2t}\left(t^2\cdot\frac{1}{2}e^{2t} - \int te^{2t}\,dt\right)$

$= e^{-2t}\left(\frac{1}{2}t^2e^{2t} - \left(t\cdot\frac{1}{2}e^{2t} - \int\frac{1}{2}e^{2t}\,dt\right)\right)$

$y = \underline{\frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{4} + Ce^{-2t}}$

$y(0) = 1$: $\quad 1 = 1/4 + C \Rightarrow C = 3/4 \Rightarrow y = \underline{\frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{4} + \frac{3}{4}e^{-2t}}$

4) a) $ty' + 2y + t = 0 \qquad (t \neq 0)$

$$y' + \frac{2y}{t} = -1 \qquad u = e^{2\ln t} = t^2$$

$$(t^2 y)' = -t^2$$

$$t^2 y = \int -t^2\,dt = -\frac{1}{3}t^3 + C$$

$$y = -\frac{t}{3} + \frac{C}{t^2}$$

b) $y' - \frac{1}{4}y = t \qquad (t > 0)$

$$\left(y e^{-1/4 t}\right)' = t e^{-1/4 t}$$

$$y = e^{1/4 t}\left(t \cdot (-4) e^{-1/4 t} - \int 1 \cdot (-4) e^{-1/4 t}\,dt\right)$$

$$= -4t - 16 + Ce^{1/4 t}$$

$$y = -4t - 16 + Ce^{1/4 t}$$

c) $y' - \frac{t}{t^2-1}y = t \qquad (t > 1)$

$$u = e^{-\frac{1}{2}\ln|t^2-1|} = (t^2-1)^{-1/2}$$

$$\left(y \cdot (t^2-1)^{-1/2}\right)' = t(t^2-1)^{-1/2}$$

$$y = \sqrt{t^2-1}\cdot\left((t^2-1)^{1/2} + C\right) = C\sqrt{t^2-1} + t^2 - 1$$

5) a) $y' = 4(y-1)(y-3)$, $y(0) = 2$

$$\frac{1}{(y-1)(y-3)} = \frac{A}{y-1} + \frac{B}{y-3}$$

$$1 = A(y-3) + B(y-1)$$

$y = 3: \ 1 = 2B \Rightarrow B = 1/2$

$y = 1: \ 1 = -2A \Rightarrow A = -1/2$

$$\int \frac{1}{(y-1)(y-3)}\,dy = \int 4\,dt$$

$$\frac{1}{2}\ln|y-3| - \frac{1}{2}\ln|y-1| = 4t + C$$

$$\ln\left|\frac{y-3}{y-1}\right| = 8t + 2C \Rightarrow \frac{y-3}{y-1} = \pm e^{8t}\cdot e^{2C} = Ke^{8t}$$

$$Ke^{8t}(y-3) = y-1 \Rightarrow y = \frac{3Ke^{8t}-1}{Ke^{8t}-1}$$

~~$e^{2t}y' - y^2 - 2y = 1$~~

$y(0)=2$:

$$2 = \frac{3K-1}{K-1} \Rightarrow 2K-2 = 3K-1$$

$$K = -1$$

$$y = \frac{-3e^{8t}-1}{-e^{8t}-1} = \frac{3e^{8t}+1}{e^{8t}+1}$$

b) $e^{2t}y' - y^2 - 2y = 1$, $y(1) = 1$

$$e^{2t}y' = y^2+2y+1 = (y+1)^2$$

$$\frac{1}{(y+1)^2}y' = e^{-2t}$$

$$-\frac{1}{(y+1)} = -\frac{1}{2}e^{-2t} + C$$

$$\frac{1}{y+1} = \frac{1}{2}e^{-2t} - C \Rightarrow y+1 = \frac{1}{\frac{1}{2}e^{-2t}-C} = \frac{2e^{2t}}{1-C\cdot 2e^{2t}}$$

$y(1)=1$:

$$\frac{1}{2} = \frac{1}{2}e^{-2} - C$$

$$C = \frac{1}{2} - \frac{1}{2}e^{-2} = \frac{1}{2}(1-e^{-2})$$

$$y+1 = \frac{2e^{2t}}{1-(1-e^{-2})e^{2t}} = \frac{2e^{2t}}{1-e^{2t}+e^{2t-2}}$$

c) $y' - \frac{t}{t^2-1}y = 0$, $y(0) = 1$

$$\frac{1}{y}y' = \frac{t}{t^2-1}$$

$$\ln|y| = \frac{1}{2}\ln|t^2-1| + C$$

$$|y| = |t^2-1|^{1/2}\cdot e^C \Rightarrow y = \pm e^C\sqrt{|t^2-1|} = K\sqrt{|t^2-1|}$$

$y(0)=1$:

$$1 = K\cdot\sqrt{1} \Rightarrow K = 1$$

$$y = \sqrt{1-t^2}$$

6) a) $y'' = t \qquad y' = \int t\,dt = \frac{1}{2}t^2 + C_1 \qquad y = \int \frac{1}{2}t^2 + C_1\,dt$

$$\underline{\underline{y = \frac{1}{6}t^3 + C_1 t + C_2}}$$

b) $y'' = e^t + t^2 \qquad y' = \int e^t + t^2\,dt = e^t + \frac{1}{3}t^3 + C_1$

$$y = \int e^t + \frac{1}{3}t^3 + C_1\,dt = \underline{\underline{e^t + \frac{1}{12}t^4 + C_1 t + C_2}}$$

7) $y'' = t^2 - t, \quad y(0) = 1, \quad y'(0) = 2$

$$y' = \frac{1}{3}t^3 - \frac{1}{2}t^2 + C_1 \Rightarrow y = \underline{\frac{1}{12}t^4 - \frac{1}{6}t^3 + C_1 t + C_2}$$

$$\left.\begin{array}{l} y(0) = 1: \quad C_2 = 1 \\ y'(0) = 2: \quad C_1 = 2 \end{array}\right\}, \qquad y = \underline{\underline{\frac{1}{12}t^4 - \frac{1}{6}t^3 + 2t + 1}}$$

8) $y'' = y' + t \quad , \; y(0) = 1, \; y'(1) = 2$

Sett $u = y'$: $\quad u' = u + t$

$$u' - u = t$$
$$(ue^{-t})' = te^{-t}$$
$$u = e^t \cdot \int te^{-t}\,dt = e^t\left(-te^{-t} - e^{-t} + C\right)$$
$$= -t - 1 + Ce^t$$

$$y' = -t - 1 + C_1 e^t$$
$$y = \underline{-\frac{1}{2}t^2 - t + C_1 e^t + C_2}$$

$\cancel{y(0)=1?\ -1 + 2C_1 = 1 \Rightarrow C_1 = 2}$

$$\left.\begin{array}{l} y'(1) = 2: \quad -2 + C_1 e = 2 \Rightarrow C_1 = 4/e \\ y(0) = 1: \quad C_1 + C_2 = 1 \Rightarrow C_2 = 1 - C_1 = 1 - 4/e \end{array}\right\} \qquad \underline{\underline{y = -\frac{1}{2}t^2 - t + 4e^{t-1} + 1 - 4e^{-1}}}$$

9).

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a) $y'' - 3y = 0$

$r^2 - 3 = 0 \quad r = \pm\sqrt{3}$ $\quad \underline{y = C_1 e^{\sqrt{3}t} + C_2 e^{-\sqrt{3}t}}$

b) $y'' + 4y' + 8y = 0$

$r^2 + 4r + 8 = 0$

$r = \dfrac{-4 \pm \sqrt{16 - 4\cdot 8}}{2} = -2 \pm 2\sqrt{-1}$ $\quad \underline{y = e^{-2t}(C_1 \cos 2t + C_2 \sin 2t)}$

c) $3y'' + 8y' = 0$

$3r^2 + 8r = 0$

$r = 0,\ r = -8/3$ $\quad y = C_1 e^{0t} + C_2 e^{-8/3t} = \underline{\underline{C_1 + C_2 e^{-8/3t}}}$

d) $4y'' + 4y' + y = 0$

$4r^2 + 4r + 1 = 0$

$r = \dfrac{-4 \pm \sqrt{16 - 4\cdot 4\cdot 1}}{2\cdot 4} = -2/4 = -1/2$ $\quad \underline{y = C_1 e^{-\frac{1}{2}t} + C_2 t e^{-\frac{1}{2}t}}$

e) $y'' + y' - 6y = 0$

$r^2 + r - 6 = 0$

$r = \dfrac{-1 \pm \sqrt{1 + 24}}{2} = \dfrac{-1 \pm 5}{2}$

$r = \underline{-3},\ r = \underline{2}$ $\quad y = \underline{C_1 e^{-3t} + C_2 e^{2t}}$

10) $y'' + y' - 6y = 7$ $\quad$ Gen. løsn: $y = \underline{\underline{-7/6 + C_1 e^{-3t} + C_2 e^{2t}}}$

Homogen: $r^2 + r - 6 = 0$

$r = -3, 2$ $\quad \Rightarrow y_h = \underline{C_1 e^{-3t} + C_2 e^{2t}}$

Part: $y = A$ $\quad$ VS: $0 + 0 - 6A$ $\quad$ HS: 7 $\quad \Big\}$ $-6A = 7$, $A = -7/6$ $\Rightarrow y_p = \underline{-7/6}$

11) $y'' - 10y' + 25y = 4$, $y(0) = 29/25$, $y(1) = 2e^5 + 4/25$

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$y'' - 10y' + 25y = 0$: $r^2 - 10r + 25 = 0$
$r = 5$
$\Rightarrow y_h = (C_1 + C_2 t)e^{5t}$

$25A = 4$
$A = 4/25$
$\Rightarrow y_p = 4/25$

Gen løsn: $y = (C_1 + C_2 t)e^{5t} + 4/25$

$y(0) = 29/25$: $C_1 + 4/25 = 29/25 \Rightarrow C_1 = 1$

$y(1) = 2e^5 + 4/25$: $(1 + C_2)e^5 + 4/25 = 2e^5 + 4/25 \Rightarrow C_2 = 1$

$$y = (1+t)e^{5t} + 4/25$$

12) $y'' + ay' + by = 0$

$\boxed{\text{Anta: } a^2 - 4b = 0}$

$r^2 + ar + b = 0$

$r = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2} = -\dfrac{a}{2}$

$r = -a/2$

$y(t) = u(t)e^{rt} = u(t)e^{-a/2\, t}$

$y' = u'e^{rt} + ue^{rt} \cdot r = (u' + ur)e^{rt}$

$y'' = (u'' + u'r)e^{rt} + (u' + ur)e^{rt} \cdot r$
$= (u'' + 2u'r + ur^2)e^{rt}$

y(t) løsning: $y'' + ay' + by = 0$

$(u'' + 2u'r + ur^2)e^{rt} + a \cdot (u' + ur)e^{rt} + be^{rt} \cdot u = 0 \quad | : e^{rt}$

$u'' + 2u'r + ur^2 + au' + aru + bu = 0$

$u'' + u' \cdot \underbrace{(2r + a)}_{= 0} + u\underbrace{(r^2 + ar + b)}_{= 0} = 0$

siden $r = -a/2$ siden r løsn. av kar.likn

$u'' = 0 \Rightarrow u = C_1 + C_2 t \Rightarrow y(t) = (C_1 + C_2 t)e^{rt}$