

$$5.) a) \text{Cov}(X, X) = E(X^2) - E(X) \cdot E(X) = \text{Var}(X)$$

$$b) \text{Cov}(X, Y) = E(XY) - E(X)E(Y) \\ = E(YX) - E(Y)E(X) = \text{Cov}(Y, X)$$

$$c) \text{Cov}(cX, Y) = E(cX \cdot Y) - E(cX) \cdot E(Y) \\ = c E(XY) - c E(X)E(Y) = c \cdot \text{Cov}(X, Y)$$

$$d) \text{Cov}(X_1 + X_2, Y) = E((X_1 + X_2)Y) - E(X_1 + X_2)E(Y) \\ = E(X_1Y + X_2Y) - (E(X_1) + E(X_2))E(Y) \\ = E(X_1Y) - E(X_1)E(Y) + E(X_2Y) - E(X_2)E(Y) \\ = \text{Cov}(X_1, Y) + \text{Cov}(X_2, Y)$$

$$6) Y = a_1 X_1 + a_2 X_2:$$

$$E(Y) = a_1 E(X_1) + a_2 E(X_2) = \begin{pmatrix} E(X_1) & E(X_2) \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \underline{\mu \cdot \underline{a}}$$

$$\text{Var}(Y) = \text{Cov}(a_1 X_1 + a_2 X_2, a_1 X_1 + a_2 X_2) \\ = a_1^2 \text{Cov}(X_1, X_1) + a_1 a_2 \text{Cov}(X_1, X_2) + a_2 a_1 \text{Cov}(X_2, X_1) \\ + a_2^2 \text{Cov}(X_2, X_2) = \underline{\underline{\underline{\underline{a^T \cdot \Sigma \cdot a}}}}}$$

$$7) Y = 5X_1 - X_2 + 2X_3$$

$$\underline{a} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \Rightarrow E(Y) = \begin{pmatrix} 14 & -2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} = \underline{28}$$

$$\text{Var}(Y) = \begin{pmatrix} 5 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} = \underline{89}$$

$$8) \begin{pmatrix} 4 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 4a_1 + 2a_2 = 6 \Rightarrow a_2 = \frac{6 - 4a_1}{2} = \underline{3 - 2a_1}$$

$$\text{Var}(Y) = \underline{a^T} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix} \underline{a} = a_1^2 + 4a_1 a_2 + 7a_2^2 = a_1^2 + 4a_1(3 - 2a_1) + 7(3 - 2a_1)^2 \\ = a_1^2 + 12a_1 - 8a_1^2 + 7(9 - 12a_1 + 4a_1^2) = 21a_1^2 - 72a_1 + 63$$

$$\text{Var}(Y)'_{a_1} = 42a_1 - 72 = 0 \Rightarrow a_1 = \frac{72}{42} = \frac{12}{7} \quad \underline{\text{min}} \quad \text{since } \text{Var}(Y)''_{a_1} = 42 > 0$$

$$\Rightarrow a_1 = \underline{\underline{\frac{12}{7}}}, \quad a_2 = 3 - 2 \cdot \frac{12}{7} = \underline{\underline{-\frac{3}{7}}}$$