FORELESNING 1 — ELE 3719

EIVIND ERIKSEN JAN 6, 2015 MATEMATIKK

Plan:

1. Om kurset
2. Vektorer og vektorregning
3. Geometri og indreprodukt
4. Lineære underrom

---

2 Vektorer og vektorregning

En vektor (n-vektor) er et ordnet n-~~tu~~ppel av tall.

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

kolonnevektor

Ex:

$$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\underline{w} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$\underline{v} = (1,2)$ svarer til kolonnevektoren $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Regning med vektorer:

i) Addisjon / subtraksjon: $\underbrace{\underline{v}+\underline{w}\,,\ \underline{v}-\underline{w}}_{\text{om } \underline{v},\underline{w} \text{ har samme størrelse}}$

Eks: $\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}4\\1\end{pmatrix}$

$\begin{pmatrix}1\\2\end{pmatrix}-\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}-2\\3\end{pmatrix}$

ii) Skalarmultiplikasjon = multiplikasjon av en vektor med et tall (skalar)

Eks: $3\cdot\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}6\\3\end{pmatrix}$

tall (skalar) vektor

$r\cdot\underline{v}$

Lineærkombinasjon: $\underbrace{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_m}_{\text{vektorer av samme størrelse}}$

$r_1\cdot\underline{v}_1 + r_2\cdot\underline{v}_2 + \ldots + r_m\cdot\underline{v}_m$

er en lineærkombinasjon av vektorene $\underline{v}_1, \ldots, \underline{v}_m$ med koeffisienter $r_1, r_2, \ldots, r_m$ (tall)

Eks:

$\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$:

Lineærkomb. av $\underline{v}_1$

$r_1 \cdot \underline{v}_1 = r_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} r_1 \\ 2r_1 \end{pmatrix}$

$\underline{v}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$:

Lineærkombinasjoner av $\{\underline{v}_1, \underline{v}_2\}$:

$$r_1 \cdot \underline{v}_1 + r_2 \cdot \underline{v}_2 = r_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + r_2 \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} r_1 + 3r_2 \\ 2r_1 - r_2 \end{pmatrix}}}$$

Eks:

$$3 \cdot \underline{x} + 2 \cdot \underline{a} = 5\underline{b}$$

$$3\underline{x} = 5\underline{b} - 2\underline{a}$$

$$\underline{x} = \frac{1}{3} \cdot (5\underline{b} - 2\underline{a})$$

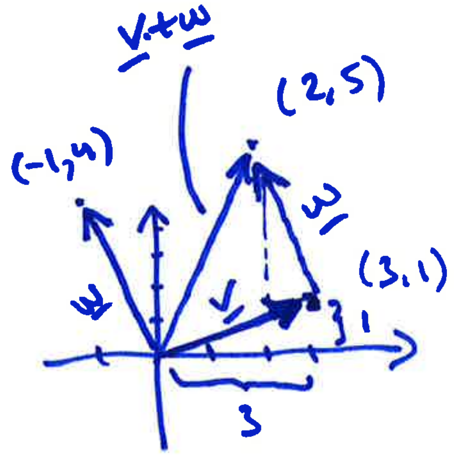
$$= \underline{\underline{\frac{5}{3} \cdot \underline{b} - \frac{2}{3} \cdot \underline{a}}}$$

Nullvektoren:

$$\underline{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Geometrisk framstilling av vektorer

2-vektorer:



$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Eks: $\underline{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ er representert ved en pil fra (0,0) til (3,1)

$$\underline{w} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

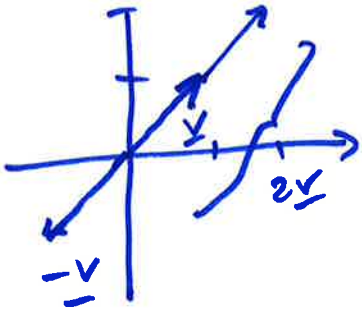
Vi tolker $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ som en forflytning

$$\underline{v} + \underline{w} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Eks: $\underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$2\underline{v} = 2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$-\underline{v} = -1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$



Generelt:

$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$:

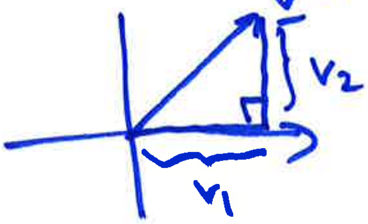
Geometrisk tolkning:

En pil fra $(0,0,\dots,0)$ til $(v_1, v_2, \dots, v_n)$

i et n-dimensjonalt koordinatsystem.

Lengden til en vektor:

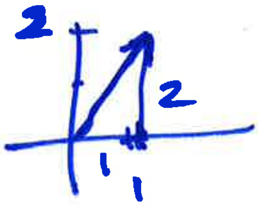
$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$



Pytagoras:

$$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2}$$

Eks: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



$$\|\underline{v}\| = \sqrt{1^2 + 2^2} = \underline{\underline{\sqrt{5}}}$$

Defn: Lengden til en n-vektor $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ til å være

$$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

Merk: $\|\underline{v}\| \geq 0$, og $\|\underline{v}\| = 0$ hvis og bare hvis $\underline{v} = \underline{0}$

3 Indreprodukt (prikkprodukt, skalarprodukt)

$\underline{v}, \underline{w}$: n-vektorer

$$\underline{v} \cdot \underline{w} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = v_1 w_1 + v_2 w_2 + \dots$$
$$\dots + v_n \cdot w_n$$

Eks: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ $\underline{v} \cdot \underline{w} = 1 \cdot 4 + 2 \cdot (-2)$
$= 4 - 4 = \underline{0}$

$$\underline{v} \cdot \underline{v} = 1 \cdot 1 + 2 \cdot 2 = \underline{5} \leftarrow \|v\| = \sqrt{5}$$

$$\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \underline{w} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\underline{v} \cdot \underline{w} = 1 \cdot 2 + 2 \cdot (-1) + 3 \cdot 3 = \underline{9}$$

$\langle \underline{v}, \underline{w} \rangle = \underline{v} \cdot \underline{w}$
skrivemåte som brukes noen ganger

Egenskaper:

i) $\underline{v} \cdot \underline{w} = \underline{w} \cdot \underline{v}$

ii) $(\underline{v_1} + \underline{v_2}) \cdot \underline{w} = \underline{v_1} \cdot \underline{w} + \underline{v_2} \cdot \underline{w}$
$(r\underline{v}) \cdot \underline{w} = r \cdot (\underline{v} \cdot \underline{w})$

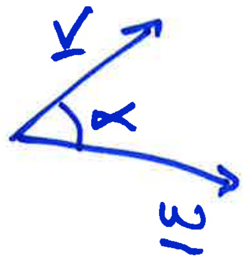
iii) $\underline{v} \cdot \underline{v} = \|\underline{v}\|^2$

$\underline{v} \cdot \underline{v} =$
$v_1 \cdot v_1 + v_2 \cdot v_2 + \dots + v_n \cdot v_n$
$= v_1^2 + v_2^2 + \dots + v_n^2$

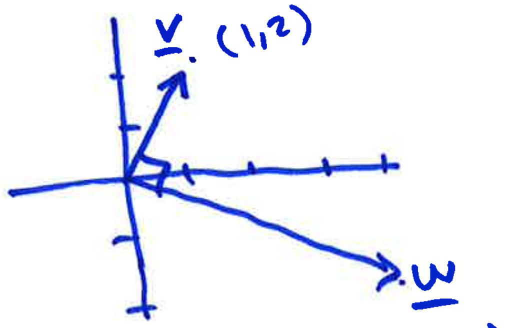
iv) $\underline{v} \cdot \underline{w} = 0$

hvis og bare hvis

$\underline{v} \perp \underline{w}$ (vektorene står normalt på hverandre, dvs er <u>ortogonale</u>)



<u>Eks:</u> $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ $\underline{v} \cdot \underline{w} = 0$



<u>Cauchy-Schwartz ulikhet:</u>

For alle n-vektorer $\underline{v}, \underline{w}$ så er

$$|\underline{v} \cdot \underline{w}| \leq \|\underline{v}\| \cdot \|\underline{w}\|$$

$$\Updownarrow$$

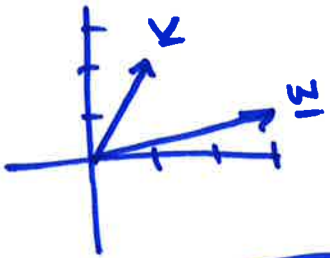
$$\frac{|\underline{v} \cdot \underline{w}|}{\|\underline{v}\| \cdot \|\underline{w}\|} \leq 1$$

$$\Updownarrow$$

$$-1 \leq \frac{\underline{v} \cdot \underline{w}}{\|\underline{v}\| \cdot \|\underline{w}\|} \leq 1$$

CS: $$-1 \le \frac{\underline{v}\cdot\underline{w}}{\|\underline{v}\|\cdot\|\underline{w}\|} \le 1$$

Es: $\underline{v} = \begin{pmatrix}1\\2\end{pmatrix}$ $\quad \underline{w} = \begin{pmatrix}3\\1\end{pmatrix}$



$\underline{v}\cdot\underline{w} = 1\cdot 3 + 2\cdot 1 = 5$

$\|\underline{v}\| = \sqrt{1^2+2^2} = \sqrt{5}$

$\|\underline{w}\| = \sqrt{3^2+1^2} = \sqrt{10}$

$$\frac{\underline{v}\cdot\underline{w}}{\|\underline{v}\|\cdot\|\underline{w}\|} = \frac{5}{\sqrt{5}\cdot\sqrt{10}}$$

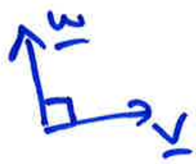
$$= \frac{\cancel{5}}{\cancel{\sqrt{5}\cdot\sqrt{5}}\cdot\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2} \simeq \underline{0.7}$$

$$\underline{\frac{\underline{v}\cdot\underline{w}}{\|\underline{v}\|\cdot\|\underline{w}\|} = \cos(\alpha)} :$$

$\cos(\alpha) = -1$

$\cos(\alpha) = 0$



$\cos(\alpha) = 1$

Korrelasjonskoeffisienten r :

x_1	y_1
x_2	y_2
x_3	y_3
$\vdots$	
x_n	y_n

datasett
(to variable
n observasjoner)

$$\bar{x} = \frac{x_1 + \dots + x_n}{n} \qquad \bar{y} = \frac{y_1 + y_2 + \dots + y_n}{n}$$

$$\underline{x} = \begin{pmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{pmatrix}$$

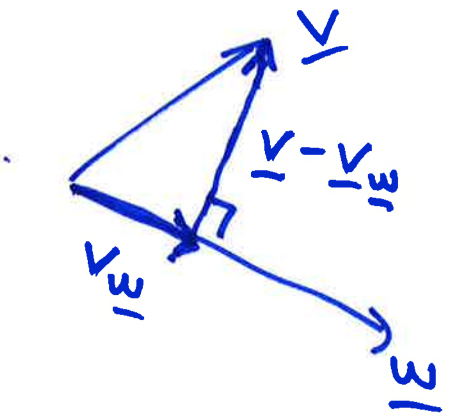
$$\underline{y} = \begin{pmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix}$$

$$\boxed{r = \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\| \cdot \|\underline{y}\|}}$$

korrelasjonskoeff.

CS: $-1 \leq r \leq 1$

Projeksjon:



$\text{proj}_{\underline{w}}(\underline{v}) = \underline{v}_{\underline{w}}$

er den vektoren som er parallell med $\underline{w}$ slik at $\underline{v}_{\underline{w}} \perp \underline{v} - \underline{v}_{\underline{w}}$

$$\underline{v}_{\underline{w}} \perp \underline{v} - \underline{v}_{\underline{w}}$$
$$\Updownarrow$$
$$\underline{v}_{\underline{w}} \cdot (\underline{v} - \underline{v}_{\underline{w}}) = 0$$

$$\underline{v}_{\underline{w}} \text{ langs } \underline{w}$$
$$\Updownarrow$$
$$\underline{v}_{\underline{w}} = k \cdot \underline{w}$$

$$k\underline{w} \cdot (\underline{v} - k\underline{w}) = 0$$

$$k \cdot (\underline{w} \cdot \underline{v}) - k^2 \cdot (\underline{w} \cdot \underline{w}) = 0 \qquad : k$$

$$\underline{v} \cdot \underline{w} - k \cdot (\underline{w} \cdot \underline{w}) = 0$$

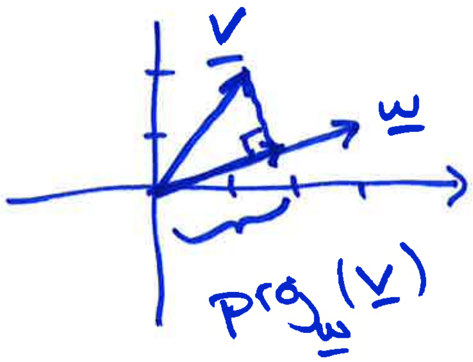
$$\underline{v} \cdot \underline{w} = k(\underline{w} \cdot \underline{w})$$

$$k = \frac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}} = \frac{\underline{v} \cdot \underline{w}}{\|\underline{w}\|^2}$$

Konklusjon:

$$\text{proj}_{\underline{w}}(\underline{v}) = \underline{v}_{\underline{w}} = \frac{\underline{v} \cdot \underline{w}}{\|w\|^2} \cdot \underline{w}$$

Eks: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \underline{w} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



$$\text{proj}_{\underline{w}}(\underline{v}) = \frac{\underline{v}\cdot\underline{w}}{\|\underline{w}\|^2}\cdot\underline{w}$$

$$= \frac{5}{(\sqrt{10})^2}\cdot\underline{w} = \frac{1}{2}\,\underline{w}$$

$$\underline{v}\cdot\underline{w} = 5$$

$$\|\underline{w}\| = \sqrt{3^2+1^2} = \sqrt{10}$$

$$= \frac{1}{2}\cdot\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix}}}$$

Vi bruker projeksjoner blant annet ifm lineær regresjon i flere variable senere i kurset.

④ Lineære underrom og lineær uavhengighet

$\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_m$: n-vektorer

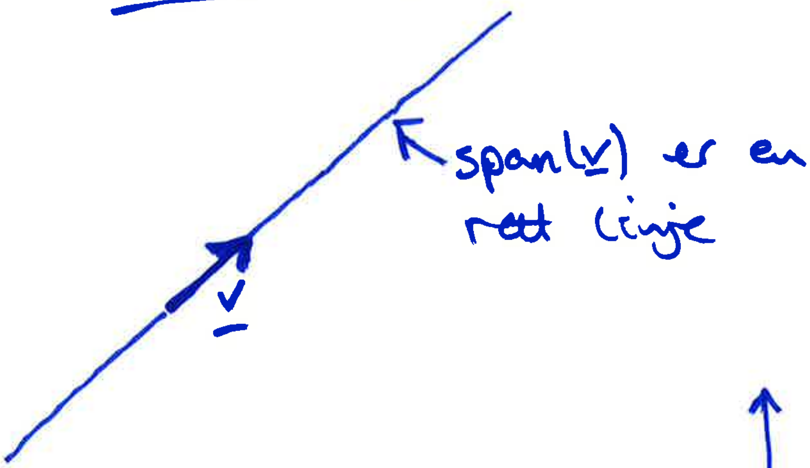
$\text{span}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_m) :=$ alle mulige lineær-kombinasjoner av vektorene

$=$ alle uttrykk som kan skrives

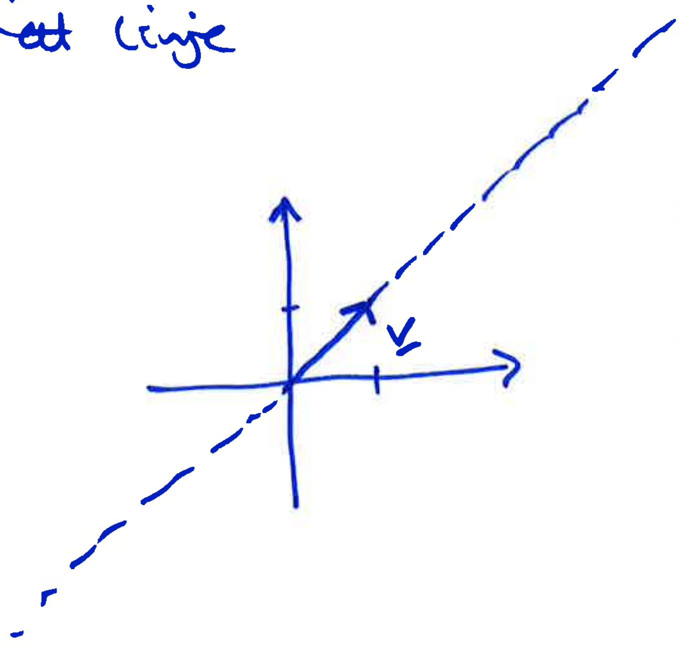
$$r_1 \cdot \underline{v}_1 + r_2 \cdot \underline{v}_2 + \ldots + r_m \cdot \underline{v}_m$$

der $r_1, r_2, \ldots, r_m$ er tall.

Eks: $\text{span}(\underline{v}) = \{ r \cdot \underline{v} \text{ der } r \text{ er et tall} \}$



$\underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

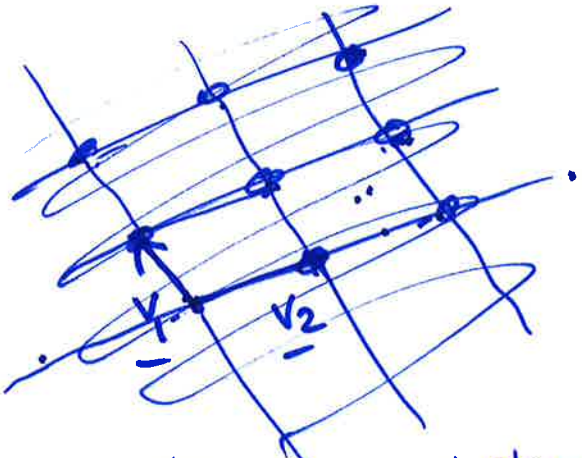


$\text{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= \left\{ r \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

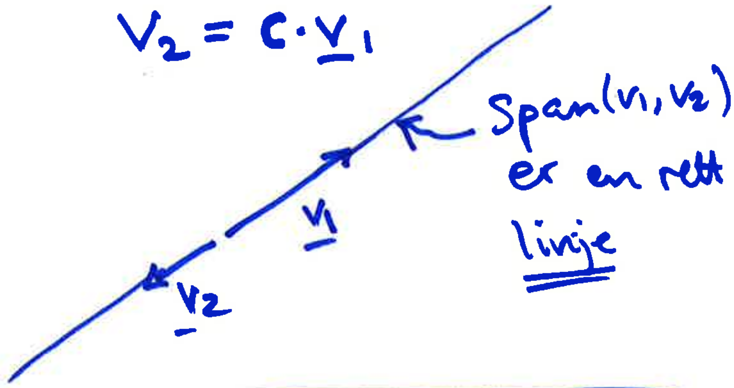
$= \left\{ \begin{pmatrix} r \\ r \end{pmatrix} \right\}$

Eks: span $(\underline{v}_1, \underline{v}_2)$



span$(\underline{v}_1,\underline{v}_2)$ er et plan

$V_2 = c\cdot\underline{v}_1$



span(v_1, v_2)

$= \{ r_1 \cdot \underline{v}_1 + r_2 \cdot \underline{v}_2 \}$

= planet som disse to vektorer utspenner

span$(\underline{v}_1, v_2)$

$= \{ r_1 \cdot \underline{v}_1 + r_2 \underline{v}_2 \}$

$= \{ r_1 \underline{v}_1 + r_2 (c\underline{v}_1) \}$

$= \{ (r_1 + r_2 c) v_1 \}$

$=$ span (v_1)

Defn: Et lineært underrom er et rom som kan skrives på formen span$\{\underline{v}_1, \ldots, \underline{v}_m\}$.

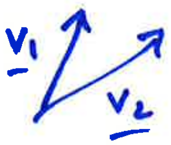
Vektorene $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_m\}$ kalles lineært avhengige hvis en (eller flere) av vektorene kan skrives som en lineær-kombinasjon av de andre; for eksempel

$$\underline{v}_1 = 4\underline{v}_2 - 2\underline{v}_4$$

Hvis ikke, så er vektorene lineært uavhengige.

Eks:

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \underline{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



I dette tilfellet er de lineært uavhengige.

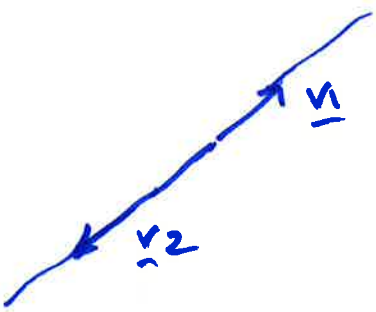
Lineært avh:

$$\left.\begin{array}{c} \underline{v}_1 = r\underline{v}_2 \\ \text{eller} \\ \underline{v}_2 = r \cdot \underline{v}_1 \end{array}\right\} \text{dette er ikke tilfellet her}$$

Eks:

$$\left.\begin{array}{l} \underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \underline{v}_2 = \begin{pmatrix} -2 \\ -4 \end{pmatrix} \end{array}\right\} \quad \underline{v}_2 = -2\underline{v}_1$$

lineært avhengige



$$W = \text{span}(\underline{v}_1, \underline{v}_2) = \text{span}(\underline{v}_1)$$

$\boxed{\text{W er en linje, ikke et plan}}$

Hvis $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_m\}$ er lineært avhengige

og $\underline{v}_1 = r_2\underline{v}_2 + r_3\underline{v}_3 + \dots + r_m\underline{v}_m$, da

er

$$\text{span}(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_m) = \text{span}(\underline{v}_2, \underline{v}_3, \dots, \underline{v}_m)$$

Defn:
Hvis $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_r\}$ er lineært uavhengige vektorer, så kalles disse vektorene en **basis** for $W = \text{span}(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_r)$. Antallet vektorer i en slik basis er **dimensjon** til W

Eks: $\underline{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$

$$\boxed{\begin{aligned} &\underline{v}_1 = c_2 \cdot \underline{v}_2 + c_3 \underline{v}_3 \\ &\text{eller} \\ &\underline{v}_2 = c_1 \cdot \underline{v}_1 + c_3 \cdot \underline{v}_3 \\ &\text{eller} \\ &\underline{v}_3 = c_1 \underline{v}_1 + c_2 \cdot \underline{v}_2 \end{aligned}}$$

$W = \text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3)$
$= \text{span}(\underline{v}_1, \underline{v}_2)$

$\underline{v}_3 = \underline{v}_1 + \underline{v}_2$

basis

W har dim 2

$c_1 \cdot \underline{v}_1 + c_2 \underline{v}_2 + c_3 \cdot \underline{v}_3 = \underline{0}$

3×3 likningssystem

$$\begin{cases} 1 \cdot \underline{v}_1 - c_2 \underline{v}_2 - c_3 \underline{v}_3 = \underline{0} \\ -c_1 \underline{v}_1 + \underline{v}_2 - c_3 \underline{v}_3 = \underline{0} \\ -c_1 \underline{v}_1 - c_2 \underline{v}_2 + \underline{v}_3 = \underline{0} \end{cases}$$