FORELESNING 10

ELE 3719

EIVIND ERIKSEN — MAR 13, 2015 — MATEMATIKK

Plan:

① Betingede fordelinger

a) diskret b) kont.

② Anvendelser

- variansen til en (vektet) sum

③ Oppgaver

Pensum:

[S] 6.9–6.11

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① Betingede fordelinger:

$$p(A \mid B) = \frac{p(A \cap B)}{p(B)} \longleftarrow \text{betinget sannsynlighet}$$

Eks: Vi kaster to terninger $\begin{cases} X = \text{ant. seksere} \\ Y = \text{summen} \end{cases}$

Y \ X	0	1	2	
2	1/36	0	0	
3	2/36	0	0	
4	3/36	0	0	
5	4/36	0	0	
6	5/36	0	0	
7	4/36	2/36	0	
8	3/36	2/36	0	5/36
9	2/36	2/36	0	
10	1/36	2/36	0	
11	0		0	
12	0	[illegible]	1/36	

X og Y er ikke uavhengige

Kan regne ut $\mathrm{Cov}(X,Y)$

$$= \underbrace{\sum_{x,y} xy \cdot p(x,y)}_{E(XY)} - E(X)\cdot E(Y)$$

$$p(X=1 \mid Y=8) = \frac{p(1,8)}{p_Y(8)} = \frac{2/36}{5/36} = \frac{2}{5}$$

Betinget fordeling:

$$p_{X|Y}(x|y) = \frac{p(X=x, Y=y)}{p(Y=y)} = p(X=x \mid Y=y)$$

$$p_{X|Y}(1|8) = \underline{\underline{2/5}}$$

Dette er en sannsynlighetstetthet i den forstand at for en vilkårlig y, så holder

i) $p_{X|Y}(x|y) \geq 0$ for alle x

ii) $\sum_{x} p_{X|Y}(x|y) = 1$

Vi kan regne ut forventningsverdien

$$E[X \mid Y=y] = \sum_{x} x \cdot p_{X|Y}(x|y)$$

Eks: $E[X \mid Y=8] = \sum_{x=0}^{2} x \cdot p_{X|Y}(x|8)$

$$= 0 \cdot p_{X|Y}(0|8) + 1 \cdot p_{X|Y}(1|8) + 2 \cdot p_{X|Y}(2|8)$$

$$= 0 + 1 \cdot 2/5 + 2 \cdot 0 = \underline{\underline{2/5}}$$

$$E[X \mid Y=7] = \text{--}$$

Variabelen gitt ved $y \longrightarrow E[X \mid Y=y]$ er en ny stokastisk variabel.

Kontinuerlig tilfelle : X, Y simultant fordelte kont. stok. variable med $f(x,y)$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$E[X \mid Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dx$$

(2) Varians til en (vektet) sum

X, Y simultant fordelte stok. variable

$\underbrace{X+Y}_{\text{Sum}}$, $\underbrace{cX+dY}_{\text{vektet Sum}}$

(c, d konst.)

$Var(X) = E(X^2) - E(X)^2$ gir

$$Var(X+Y) = E[(X+Y)^2] - (E[X+Y])^2$$

$$= E[X^2 + 2XY + Y^2] - (E(X)+E(Y))^2$$

$$= \underline{E(X^2)} + 2E(XY) + \underline{E(Y^2)} \underline{- E(X)^2} - 2E(X)E(Y) - \underline{E(Y)^2}$$

$$= Var(X) + Var(Y) + 2(Cov(X,Y))$$

$$= Var(X) + 2Cov(X,Y) + Var(Y)$$

$$Var(cX+dY) = c^2 \cdot Var(X) + 2cd\, Cov(X,Y) + d^2 \cdot Var(Y)$$

$$= (c \;\; d) \cdot \begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix}$$

$$= (c \;\; d) \cdot \underbrace{\begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{pmatrix}}_{\text{Kovariansmatrisen}} \cdot \begin{pmatrix} c \\ d \end{pmatrix}$$

Kovariansmatrisen:

Anta $X_1, X_2, \ldots, X_n$ er simultant fordelte stokastiske variable.

<u>Kovariansmatrisen</u> er da

$$C = \begin{pmatrix} \mathrm{Cov}(X_i, X_j) \end{pmatrix}$$

← Symmetrisk matrise med varianse på diagonalen

$$= \begin{pmatrix} \mathrm{Cov}(X_1,X_1) & \mathrm{Cov}(X_1,X_2) & \cdots & \mathrm{Cov}(X_1,X_n) \\ \mathrm{Cov}(X_2,X_1) & \mathrm{Cov}(X_2,X_2) & \cdots & \cdots \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

<u>Husk:</u>

$$\mathrm{Cov}(X_i,X_i) = \underbrace{E(X_i \cdot X_i)}_{X_i^2} - \underbrace{E(X_i)\cdot E(X_i)}_{E(X_i)^2}$$

$$= \mathrm{Var}(X_i)$$

$$\mathrm{Cov}(X_i,X_j) = E(X_i \cdot X_j) - E(X_i)\cdot E(X_j)$$

$$\parallel$$

$$\mathrm{Cov}(X_j,X_i) = E(X_j X_i) - E(X_j)\cdot E(X_i)$$

<u>Forventningsvektoren</u> er

$$\underline{\mu} = \begin{pmatrix} E(X_1) & E(X_2) & \ldots & E(X_n) \end{pmatrix}$$

Vi kan tenke oss at vi har n verdipapirer

$$S_1 \quad S_2 \quad \cdots \quad S_n$$

som vi kan investere i, og X_i er avkastningen til S_i.

$$Y = w_1 X_1 + w_2 X_2 + \cdots + w_n X_n$$

$$(w_1 + w_2 + \cdots + w_n = 1)$$

$$E(Y) = E(w_1 X_1 + \cdots + w_n X_n)$$

$$= w_1 E(X_1) + w_2 E(X_2) + \cdots + w_n E(X_n)$$

$$= \underline{\mu} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \left(E(X_1)\ E(X_2) \cdots E(X_n) \right) \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

$$E(Y) = \underline{\mu} \cdot \underline{w}$$

$$\mathrm{Var}(Y) = \mathrm{Var}(w_1 X_1 + w_2 X_2 + \cdots + w_n X_n)$$

$$= (w_1\ w_2 \cdots w_n) \cdot \begin{pmatrix} \mathrm{Cov}(X_i, X_j) \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

$$= \underline{w}^T \cdot C \cdot \underline{w}$$

Anta:

$$\underline{\mu} = (4 \quad -2 \quad 3)$$

$$C = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix}$$

Forventet avkastning:

$$\underline{\mu} \cdot \underline{w} = (4 \quad -2 \quad 3) \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 4w_1 - 2w_2 + 3w_3$$

Varians:

$$\underline{w}^T \cdot C \cdot \underline{w} = (w_1 \; w_2 \; w_3) \cdot \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

Dersom forventet avkastning er r, hvordan ka vi velge porteføljen slik at $Var(Y)$ blir minst mulig.

$$\min \; \underline{w}^T C \underline{w} \quad \text{når} \quad \begin{cases} \underline{\mu} \cdot \underline{w} = r \\ w_1 + w_2 + w_3 = 1 \end{cases}$$

$$L = \underline{w}^T C \underline{w} - \lambda_1 \cdot (\underline{\mu} \cdot \underline{w}) - \lambda_2 \cdot (w_1 + w_2 \; w_3)$$

$$= 2w_1^2 + 4w_1 w_2 + 4w_1 w_3 + 3w_2^2 + 2w_2 w_3 + 5w_3^2$$
$$- \lambda_1 \cdot (4w_1 - 2w_2 + 3w_3) - \lambda_2 \cdot (w_1 + w_2 + w_3)$$

$$L'_{w_1} = 4w_1 + 4w_2 + 4w_3 - 4\lambda_1 - \lambda_2 = 0$$

$$L'_{w_2} = 4w_1 + 6w_2 + 2w_3 + 2\lambda_1 - \lambda_2 = 0$$

$$L'_{w_3} = 4w_1 + 2w_2 + 10w_3 - 3\lambda_1 - \lambda_2 = 0$$

$4w_1 - 2w_2 + 3w_3 = r$

$w_1 + w_2 + w_3 = 1$

$$\frac{\partial L}{\partial \underline{w}} = 2C \cdot \underline{w} - \lambda_1 \cdot \underline{\mu}^T - \lambda_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2C \cdot \underline{w} = \lambda_1 \cdot \mu^T + \lambda_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\underline{w} = \frac{1}{2} C^{-1} \cdot \left(\lambda_1 \underline{\mu}^T + \lambda_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

Bibetingelser: $\underline{\mu} \cdot \underline{w} = r$

$w_1 + w_2 + w_3 = 1$

Vi finner en løsning w_1, w_2, w_3 som gir minimum.

Eks: $r=10$: $w_1 = \frac{277}{121} \simeq 2.29$ $\quad w_2 = \frac{-114}{121} \simeq -0.94$ $\quad w_3 = \frac{-42}{121} \simeq -0.35$

$$Var(Y) = \frac{314}{121} \simeq 2.595$$

8.3 del 2: $\underset{\substack{\text{normalfordelt} \\ (\mu, \sigma)}}{X} \sim Z = \frac{X-\mu}{\sigma} = \frac{1}{\sigma}(X-\mu)$

$E(Z) = 0$

$Var(Z) = 1$

$E\left(\frac{1}{\sigma}(X-\mu)\right) = \frac{1}{\sigma}(E(X)-\mu)$

$= \frac{1}{\sigma}\cdot(\mu-\mu) = \underline{0}$

Vis at Z er normalfordelt:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$f_Z(z)$ skal bli $\frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$$F_Z(b) = p(Z \le b) = p\left(\frac{X-\mu}{\sigma} \le b\right)$$

$$= p(X \le b\sigma + \mu) = F_X(b\sigma+\mu)$$

$$= \int_{-\infty}^{b\sigma+\mu} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$\boxed{\begin{array}{l} \underline{\text{substitusjon:}} \\ z = \frac{x-\mu}{\sigma} \\ dz = \frac{1}{\sigma}\cdot dx \end{array}}$$

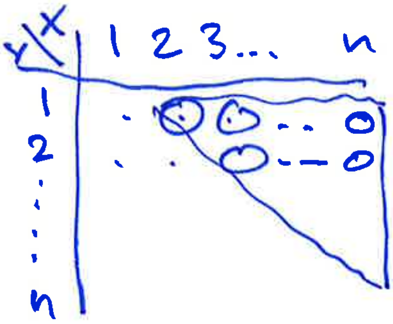
$$= \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \Rightarrow \underline{f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}}$$

normalfordelt med $\mu = 0, \sigma = 1$.

$$z^2 = \frac{(x-\mu)^2}{\sigma^2}$$

6.2.2. $p(x,y) = n^{-2} = \frac{1}{n^2}, \quad 1 \le x,y \le n$

$$p(X>Y) = p(X>Y=1) + p(X>Y=2) + \ldots$$



$$= \frac{1}{n^2} \cdot \left((n-1) + (n-2) + \ldots + 1 + 0\right)$$

$$= \frac{1}{\cancel{n^2}} \cdot \left(\frac{\cancel{n}}{2} \cdot (n-1)\right) = \underline{\underline{\frac{1}{2} \cdot \frac{n-1}{n}}}$$

$$p(X=Y) = p(X=Y=1) + p(X=Y=2) + \ldots$$

$$= \frac{1}{n^2} \cdot n = \underline{\underline{\frac{1}{n}}}$$

$$p(X>Y) = \left(1 - \frac{1}{n}\right) \cdot \frac{1}{2} = \underline{\underline{\frac{1}{2} \cdot \frac{n-1}{n}}}$$

6.3.3: $F(x,y) = 1 - e^{-xy}, \quad 0 \le x,y < \infty$

Er dette en kumulativ fordeling?

$$f(x,y) = F''_{xy}$$

$$= \left(y e^{-xy}\right)'_y$$

$$= 1 \cdot e^{-xy} + (-x) y e^{-xy}$$

$$= (1-xy) e^{-xy} \not\ge 0 \text{ for alle } x,y \ge 0$$

Ikke tetthet/fordelingsfunksjon.

Krav:

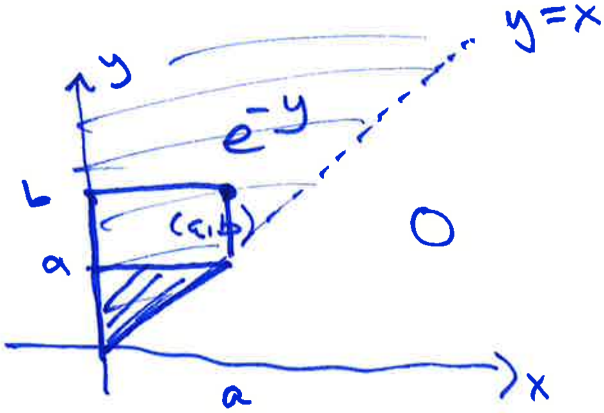
i) $f(x,y) \ge 0$

ii) $\iint f(x,y)\,dy\,dx = 1$

6.3.1.

$f(x,y) = e^{-y}, \quad 0 \le x < y < \infty$

Hva er $F(x,y)$?



$F(a,b) = p(X \le a, Y \le b)$

Anta: $b > a$

$$F(a,b) = \left\{ \int_0^{y}\int_0^{a} e^{-y}\,dy\,dx + \int_0^{a}\int_a^{b} e^{-y}\,dy\,dx \right\} = \left\{ \int_0^{a}\int_0^{y} e^{-y}\,dx\,dy + \int_0^{a}\int_a^{b} e^{-y}\,dy\,dx \right\}$$

$$= \int_0^a \left[x e^{-y}\right]_0^y dy + \int_0^a \left[-e^{-y}\right]_a^b dx$$

$$= \int_0^a y e^{-y}\,dy + \int_0^a \left(-e^{-b} + e^{-a}\right) dx$$

$$= \left[-ye^{-y} - e^{-y}\right]_0^a + a\cdot\left(e^{-a} - e^{-b}\right)$$

$$= \cancel{-ae^{-a}} - e^{-a} + 1 + \cancel{ae^{-a}} - ae^{-b}$$

$$= \underline{\underline{1 - e^{-a} - ae^{-b}}}, \quad b \ge a$$

$$\int y e^{-y}\,dy = \int u v'\,dy = uv - \int u' v\,dy$$

$u = y \quad v' = e^{-y}$

$u' = 1 \quad v = -e^{-y}$

$$= -ye^{-y} + \int e^{-y}\,dy$$

$$= \underline{-ye^{-y} - e^{-y} + C}$$

5.12.5 Vis at $E(x)^2 \leq E(x^2)$ når $E(x)$ er endelig.

$$\Updownarrow$$

$$0 \leq E(x^2) - E(x)^2$$

$$0 \leq Var(x)$$

Nå

$$Var(x) = E\left[(X-\mu)^2\right] \geq 0$$

siden $(X-\mu)^2 \geq 0$

$$E\left[(x-\mu)^2\right] = \sum_x (x-\mu)^2 \cdot p(x) \geq 0$$

5.12.6:

c) $f(x) = \underline{c \cdot e^{-x^2+4x}}$ for alle x

i) $f(x) \geq 0$ ok $c \geq 0$

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$?

Må prøve å skrive som $\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$e^{-x^2+4x} = e^{-(x-2)^2+4} = e^4 \cdot e^{-(x-2)^2}$$

$f(x) = c\, e^4\, e^{-(x-2)^2}$

$\mu = 2$

$\sigma = 1/\sqrt{2}$

$c \cdot e^4 = \frac{1}{\sqrt{\cancel{2}\pi}} \cdot \frac{1}{1/\cancel{\sqrt{2}}}$

$c = e^{-4} \cdot \frac{1}{\sqrt{\pi}}$

Konklusjon:

Hvis $\underline{c = e^{-4} \cdot \frac{1}{\sqrt{\pi}}}$, så er X normalfordelt.

med $\underline{\mu = 2}$, $\underline{\sigma = 1/\sqrt{2}}$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$