FORELESNING 12 — ELE 3719 MATEMATIKK

EIVIND ERIKSEN — MAR 24, 2015

Plan:

1. Lineære første ordens diff. likninger
2. Stabilitet
3. Systemer av diff. likninger

Pensum: [DIFF] 3

1. Lineære første ordens diff. likninger

Defn: En diff. likning av første orden er *lineær* hvis den skrives som

$$\boxed{y' + a(t)\cdot y = b(t)} \iff y' = \underbrace{b(t) - a(t)\cdot y}_{F(y,t)}$$

der $a(t), b(t)$ er uttrykk i t. (lineært i y)

Eks:

$$y' + 2y = 0$$

(separabel, lineær)

$$y' = -2y = y\cdot(-2)$$

$$y' + \underbrace{2t}_{a(t)}y = \underbrace{e^{t-t^2}}_{b(t)}$$

(lineær)

$$y' = e^{t-t^2} - 2ty$$

Navn: En lineær diff. likn. er *homogen* om $b(t) = 0$, og *inhomogen* ellers.

Den har konstante koeff. hvis $a(t)=a$, $b(t)=b$ er to konstanter.

Tilfelle I: Homogen med konstante koeff.

$$y' + ay = 0 \qquad (a \text{ konstant})$$

Separabel:

$$y' = -ay$$

$$\frac{1}{y}y' = -a$$

$$\int \frac{1}{y}\,dy = \int -a\,dt$$

$$\ln|y| = -at + C$$

$$|y| = e^{-at+C}$$

$$y = \pm e^{-at}\cdot e^{C}$$

$$y = K\cdot e^{-at}$$

Karakteristisk likning:

$$r + a = 0$$

$$r = -a$$

$$\Downarrow$$

$$\boxed{y = C\cdot e^{-at}}$$

Hvorfor fungerer dette?

Vi "gjetter" løsningen $y = e^{rt}$:

$$y' + ay = 0$$

$$(e^{rt}\cdot r) + a\cdot(e^{rt}) = 0$$

$$\cancel{e^{rt}}\cdot(r+a) = 0$$

$$\boxed{r + a = 0}$$

Tilfelle 2: Inhomogen med konstante koeffisienter

$$y' + ay = b \qquad (a, b \text{ konstanter})$$

(Kan løses som separabel diff. likn.)

"Superposition principle":

I en lineær diff. likning så er den generelle løsn $y = y(t)$ gitt ved

$$y = y_h + y_p$$

y_h: generell løsn. av den homogene likn. $y' + ay = 0$, dvs

$$y_h = C \cdot e^{-at}$$

fra tilfelle I

y_p: en partikulær løsn. av $y' + ay = b$

Eks: $y' - 2y = 6$

$$y = y_h + y_p = C \cdot e^{2t} + (-3) = \underline{\underline{C \cdot e^{2t} - 3}}$$

$\underline{y_h}$: $y' - 2y = 0$

Kar. likn: $r - 2 = 0$

$r = 2$

$y_h = C \cdot e^{2t}$

$\underline{y_p}$: $y' - 2y = 6$

$\underline{y_p = -3}$

Generelt: $y' + ay = b$

$$y = y_h + y_p = \underline{\underline{Ce^{-at} + \frac{b}{a}}}$$

y_h: $y' + ay = 0$

$r + a = 0$

$r = -a$

$y_h = \underline{C \cdot e^{-at}}$

y_p: $y' + ay = b$

$y_p = \frac{b}{a}$

Hvorfor fungerer "superposition principle": $y = y_h + y_p$?

Lineær diff.likning: $\underbrace{y' + a(t)y}$ $= b(t)$

Definer funksjonal: $J(y) = y' + a(t) \cdot y$

Eks: $J(y) = y' + 2t \cdot y$ med $a(t) = 2t$

$J(t) = 1 + 2t \cdot t = 1 + 2t^2$

$J(1+t) = 1 + 2t \cdot (1+t) = 1 + 2t + 2t^2$

Egenskaper ved $J(y) = y' + a(t) \cdot y$

i) $J(y_1 + y_2) = J(y_1) + J(y_2)$

ii) $J(c \cdot y_1) = c \cdot J(y_1)$ (for c konst)

$J(y_1 + y_2) = (y_1 + y_2)' + a(t) \cdot (y_1 + y_2)$
$= y_1' + y_2' + a(t)y_1 + a(t)y_2$
$= J(y_1) + J(y_2)$

Vet at $J(y_h) = 0$, $J(y_p) = b(t)$.

Da følger det at:

$J(y_h + y_p) = J(y_h) + J(y_p) = 0 + b(t) = b(t) \Rightarrow y_h + y_p$ er løsn.

y løsn $\Rightarrow J(y) = b(t) \Rightarrow J(y - y_p) = b(t) - b(t) = 0$

$\Rightarrow y - y_p = y_h \Rightarrow y = y_h + y_p$.

Tilfelle 3: Ikke-konstante koeffisienter

$$y' + a(t)\,y = b(t)$$

$a(t), b(t)$ ikke konst.

$\Downarrow$

$y = C \cdot e^{-at} + \frac{b}{a}$

fungerer ikke lenger

Vi kan bruke $y = y_h + y_p$, men det blir vanskeligere å finne y_h og y_p.

Metode: Integrerende faktor

Eks:

$$y' + 2t\,y = e^{t-t^2} \quad \Big| \cdot u \quad u = u(t) \text{ (integrerende faktor)}$$

$$uy' + 2t \cdot u \cdot y = u \cdot e^{t-t^2}$$

$$uy' + u'y = ue^{t-t^2}$$

$$(u \cdot y)' = u \cdot e^{t-t^2}$$

$$uy = \int u \cdot e^{t-t^2}\,dt$$

$$y = \frac{1}{u}\int u e^{t-t^2}\,dt$$

Vi må velge $u = u(t)$ slik at $a(t) \cdot u = u'$

$2t \cdot u = u'$

Vi kan velge:

$$\boxed{u = e^{\int a(t)dt}} = e^{\int 2t dt} = e^{t^2}$$

$$(e^{t^2})' = e^{t^2} \cdot 2t$$

$$y' + \underline{2t} \cdot y = e^{t-t^2}$$

Integrerende faktor:

$$u = e^{\int a(t)dt} = e^{t^2}$$

$$e^{t^2} \cdot y' + 2t \cdot e^{t^2} \cdot y = e^{t^2} \cdot e^{t-t^2}$$

$$(y \cdot e^{t^2})' = e^t$$

$$y \cdot e^{t^2} = \int e^t \, dt = e^t + C$$

$$y = \frac{e^t + C}{e^{t^2}} = \underline{\underline{e^{t-t^2} + C \cdot e^{-t^2}}}$$

Generell metode: Integrerende faktor

$$y' + a(t) \cdot y = b(t) \quad | \cdot u$$

$u(t) = e^{\int a(t)dt}$ er int. faktor

$$(u \cdot y)' = b(t) \cdot u$$

$$u \cdot y = \int b(t) \cdot u \, dt$$

$$y = \frac{1}{u} \cdot \int (b(t) \cdot u(t)) \, dt$$

$$y = e^{-\int a(t)dt} \cdot \int b(t) \cdot e^{\int a(t)dt} \, dt$$

Eks:

$$y' = y + t$$

$$y' - y = t \qquad a(t) = -1 \qquad b(t) = t$$

Inf. faktor:

$$y' - y = t$$

$$a(t) = -1$$
$$\int a(t)\,dt = -t + C$$
$$u = e^{-t}$$

$$(e^{-t} \cdot y)' = te^{-t}$$

$$e^{-t} y = \int \underbrace{te^{-t}}_{u \cdot v'} dt = -te^{-t} - \int 1 \cdot (-e^{-t}) dt$$

delvis:

$u = t \quad v' = e^{-t}$

$u' = 1 \quad v = -e^{-t}$

$$e^{-t} y = -te^{-t} + (-e^{-t}) + C$$

$$e^{-t} y = -te^{-t} - e^{-t} + C$$

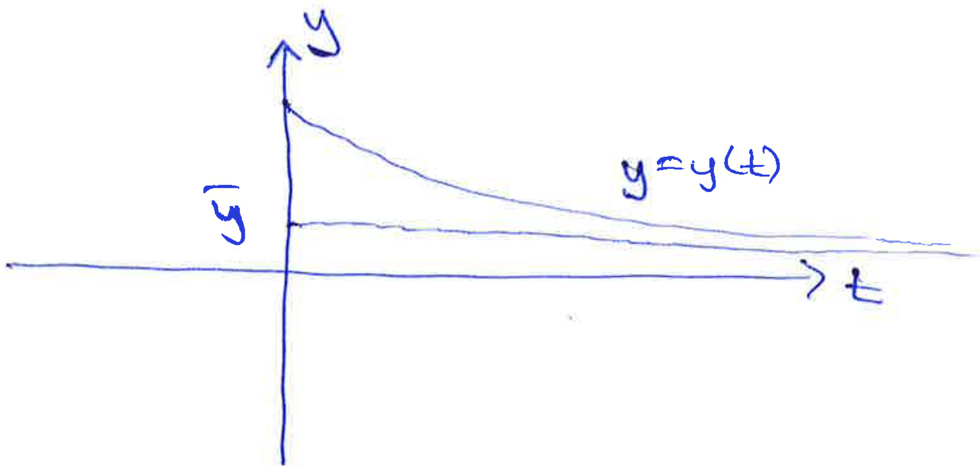
$$y = e^{t} \cdot (-te^{-t} - e^{-t} + C)$$

$$y = \underline{\underline{-t - 1 + Ce^{t}}}$$

② Stabilitet

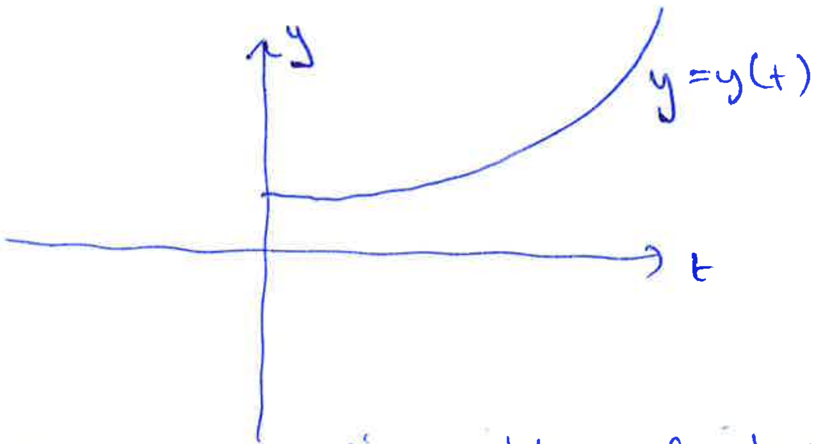
Løsningen av en lineær diff.likning

$$y = y_h + y_p$$
$$= C \cdot e^{-at} + \frac{b}{a} \quad (\text{konst.-koeff.})$$



La $\bar{y} = \lim_{t \to \infty} y(t)$ hvis grenseverdien eksisterer.
Da kalles diff.likn. stabil, med langsiktig likevektstilstand $\bar{y}$.

$(\bar{y} = \infty)$



Hvis grenseverdien ikke eksisterer, er likn. ustabil.

Eks:

$$y' + a \cdot y = b$$

$$y = C \cdot e^{-at} + \frac{b}{a} \quad , (a \neq 0)$$

$$\bar{y} = \lim_{t \to \infty} y(t) = \lim_{t \to \infty} C \cdot e^{-at} + \frac{b}{a}$$

$e^{-at} \to 0$ hvis $a > 0$

$e^{-at} \to \infty$ hvis $a < 0$

$$= \begin{cases} \bar{y} = C \cdot 0 + \frac{b}{a} = \frac{b}{a}, & a > 0 \quad \leftarrow \text{stabil} \\ \bar{y} = \infty \cdot C + \frac{b}{a} & a < 0 \quad \leftarrow \text{ustabil} \end{cases}$$

Eks:

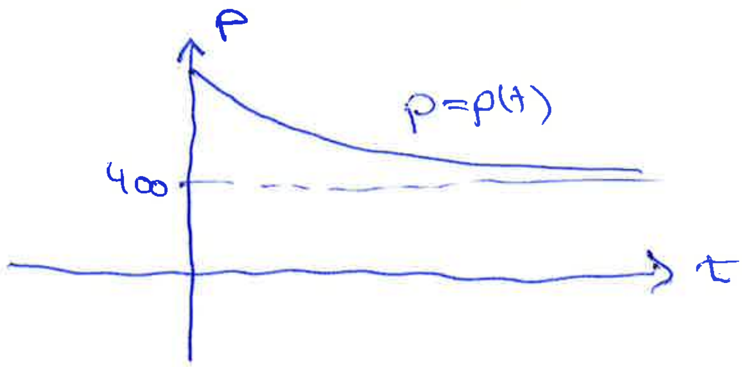
$$p' = 2000 - 5p$$

$$p' + 5p = 2000$$

$$p = \underline{C \cdot e^{-5t} + 400}$$

$$\Big\downarrow t \to \infty$$

$$\bar{p} = \lim_{t \to \infty} \left(C \underbrace{e^{-5t}}_{\downarrow \, 0} + 400 \right) = C \cdot 0 + 400 = \underline{400}$$



Stabil med $\bar{p} = 400$

Diff.likningen kalles <u>globalt asymptotisk</u> <u>stabil</u> hvis $\bar{y}$ er uavhengig av C.

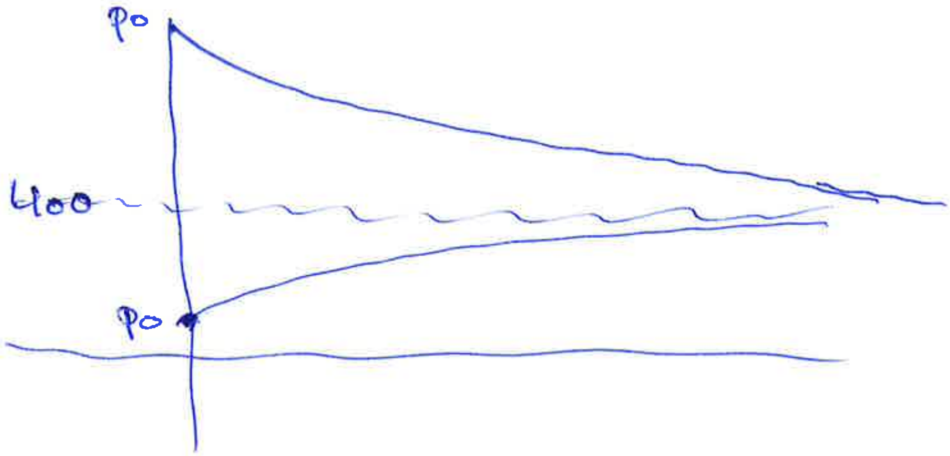
<u>Eks:</u> $p' = 2000 - 5p$

$$p = C \cdot e^{-5t} + 400$$

$$\bar{p} = 400 \quad (\text{for alle } C)$$

$$p(0) = C \cdot e^{0} + 400$$

$$p(0) = C + 400$$



globalt asymptotisk stabil

③ Systemer av lineære diff.likn.

Eks: $\cancel{x}' = x + 2y$ $\qquad x'(t) = x(t) + 2y(t)$

$y' = 4x - y$ $\qquad y'(t) = 4x(t) - y(t)$

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} : \quad \underline{\underline{x}' = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix} \cdot \underline{x}}}$$

Eks: $\underline{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \underline{x}$ $\qquad x' = 2x$

$\qquad y' = \quad -y$

$y' = -y$

$y' + y = 0$

$y = \underline{C_2 \cdot e^{-t}}$

$x' = 2x \qquad \underline{x' - 2x = 0}$

lin. homogen

$x = \underline{C_1 \cdot e^{2t}}$

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \underline{\underline{\begin{pmatrix} C_1 e^{2t} \\ C_2 \cdot e^{-t} \end{pmatrix}}}$$

Eks: $\underline{x}' = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix} \underline{x}$ $A = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$

Diagonaliserer A:

Egenverdier: $\begin{vmatrix} 1-\lambda & 2 \\ 4 & -1-\lambda \end{vmatrix} = 0$

$$\lambda^2 - 0 \cdot \lambda + (-9) = 0$$

$$\lambda^2 - 9 = 0$$

$$\underline{\lambda_1 = 3} \quad \underline{\lambda_2 = -3}$$

Egenvektorer:

$\underline{\lambda = 3}$: $\begin{pmatrix} 1-3 & 2 \\ 4 & -1-3 \end{pmatrix} \underline{x} = \underline{0}$

$\begin{pmatrix} -2 & 2 \\ \cancel{4} & \cancel{-4} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\underline{x} = \underline{\underline{t \begin{pmatrix} 1 \\ 1 \end{pmatrix}}}$

$\underline{\lambda = -3}$: $\begin{pmatrix} 4 & 2 \\ \cancel{4} & \cancel{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\underline{x} = \underline{\underline{t \begin{pmatrix} 1 \\ -2 \end{pmatrix}}}$

$P^{-1}AP = D$ når $D = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$ $P = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$

Diff. likn. system:

$$\underline{x}' = A \cdot \underline{x} \qquad P^{-1}AP = D$$

$$\underline{x}' = PDP^{-1} \cdot \underline{x} \qquad A = PDP^{-1}$$

$$P^{-1}\underline{x}' = DP^{-1}\underline{x}$$

Setter $\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix} = P^{-1} \cdot \underline{x} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}^{-1} \cdot \underline{x} = \frac{1}{-3} \cdot \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} \underline{x}$

$$\underline{u}' = P^{-1} \cdot \underline{x}' \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2/3\,x + 1/3\,y \\ 1/3\,x - 1/3\,y \end{pmatrix}$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 2/3\,x' + 1/3\,y' \\ 1/3\,x' - 1/3\,y' \end{pmatrix} = P^{-1} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\Downarrow$$

$$\underline{u}' = D \cdot \underline{u}$$

$$u' = 3u \qquad D = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$$

$$v' = \quad -3v$$

$$\left.\begin{aligned} u &= c_1 \cdot e^{3t} \\ v &= c_2 \cdot e^{-3t} \end{aligned}\right\} \quad \underline{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} c_1 \cdot e^{3t} \\ c_2 \cdot e^{-3t} \end{pmatrix}$$

$$\underline{x} = P\underline{u} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} c_1 e^{3t} \\ c_2 e^{-3t} \end{pmatrix}$$

$$\boxed{\underline{x} = c_1 \cdot \underline{v_1} \cdot e^{\lambda_1 t} + c_2 \cdot \underline{v_2} \cdot e^{\lambda_2 t}} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{3t} + c_2 e^{-3t} \\ c_1 e^{3t} - 2c_2 e^{-3t} \end{pmatrix}$$

$$= c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot e^{3t} + c_2 \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t}$$