EKSTRA FORELESNING

ELE3719

EIVIND ERIKSEN   05 JUN 2015   MATEMATIKK

Plan: Repetisjon og gjennomgang av prøve-eksamen (05/2015)

Generelle kommentarer

*) Alle svar skal begrunnes.

*) Prioriter tidsbruk.

Oppgave 1:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

a) $|A| = 1\cdot(1\cdot 2 - 4\cdot 4) + 2\cdot(0\cdot 4 - 1\cdot 2) = -14 - 4$
$= \underline{\underline{-18}}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{vmatrix}$$

$= (1-\lambda)\cdot\left[(1-\lambda)(2-\lambda) - 4\cdot 4\right] + 2\cdot(\cancel{0\cdot 4} - 2\cdot(1-\lambda))$

$= (1-\lambda)\cdot(\lambda^2 - 3\lambda + 2 - 16) - 4(1-\lambda)$

$= (1-\lambda)(\lambda^2 - 3\lambda - 14) - 4(1-\lambda)$

$= (1-\lambda)\cdot\left[\lambda^2 - 3\lambda - 14 - 4\right]$

$= \underline{\underline{(1-\lambda)\cdot(\lambda^2 - 3\lambda - 18)}} = -\lambda^3 + 4\lambda^2 + 15\lambda - 18$
$= (1-\lambda)(\lambda-6)(\lambda+3)$

$\lambda^2 - 3\lambda - 18 = 0$

$\lambda = \dfrac{3 \pm \sqrt{3^2 - 4\cdot(-18)}}{2}$

$= \dfrac{3 \pm \sqrt{81}}{2} = 6, -3$

b) Egenverdier:

$|A - \lambda I| = 0$

$(1-\lambda)(\lambda-6)(\lambda+3) = 0$

$\lambda_1 = 1 \quad \lambda_2 = 6 \quad \lambda_3 = -3$

Siden $\lambda_1 = 1, \lambda_2 = 6 > 0$

men $\lambda_3 = -3 < 0$,

så er A <u>indefinitt</u>.

<u>Definitthet</u> til en symm. matrise A med egenv. $\lambda_1, \lambda_2, \dots, \lambda_n$:

$\lambda_1, \dots, \lambda_n > 0$: A pos. defn.

$\lambda_1, \dots, \lambda_n \geq 0$: A pos. semidefn.

$\lambda_1, \dots, \lambda_n < 0$: A neg. defn.

$\lambda_1, \dots, \lambda_n \leq 0$: A neg. semidefn.

både pos. og neg. egenverdier : A indefinitt

c) Finn P slik at $P^{-1}AP = D$ er diagonal.

Må finne egenvektorer $\underline{v}_1, \underline{v}_2, \underline{v}_3$ for A, da er $P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ diagonal, med $A\underline{v}_1 = \lambda_1 \underline{v}_1$, $A\underline{v}_2 = \lambda_2 \underline{v}_2$, $A\underline{v}_3 = \lambda_3 \underline{v}_3$, siden egenverdiene alle er forskjellige, med $P = (\underline{v}_1 | \underline{v}_2 | \underline{v}_3)$.

<u>Egenvektorer:</u>

<u>$\lambda = 1$:</u> $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 6 & 4 \\ 2 & 4 & 1 \end{pmatrix} \underline{x} = \underline{0}$ $\qquad \begin{pmatrix} 2 & 4 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{pmatrix} \begin{matrix} \\ ]-2 \\ \end{matrix} \to \begin{pmatrix} 2 & 4 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

$$\begin{aligned} 2x + 4y + z &= 0 \\ z &= 0 \end{aligned} \qquad \left.\begin{aligned} x &= -2y \\ z &= 0 \end{aligned}\right\} \quad \underline{x} = \begin{pmatrix} -2y \\ y \\ 0 \end{pmatrix} = y \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

y fri

$B = (\underline{v}_1 \mid \underline{v}_2 \mid \underline{v}_3)$ ortogonal

$$\Updownarrow$$

$$\begin{pmatrix} \underline{v}_1 \cdot \underline{v}_1 & \underline{v}_1 \cdot \underline{v}_2 & \underline{v}_1 \cdot \underline{v}_3 \\ \underline{v}_2 \cdot \underline{v}_1 & \underline{v}_2 \cdot \underline{v}_2 & \underline{v}_2 \cdot \underline{v}_3 \\ \underline{v}_3 \cdot \underline{v}_1 & \underline{v}_3 \cdot \underline{v}_2 & \underline{v}_3 \cdot \underline{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Updownarrow$$

$v_i \cdot v_i = 1$

$v_i \cdot v_j = 0$ når $i \neq j$

$$P = \begin{pmatrix} -2 & 2 & -1 \\ 1 & 4 & -2 \\ 0 & 5 & 2 \end{pmatrix} \quad \underline{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \quad \underline{v}_3 = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$$

$\underline{v}_1 \cdot \underline{v}_1 = (-2)\cdot(-2) + 1\cdot 1 + 0\cdot 0 = \underline{5}$

$\underline{v}_1 \cdot \underline{v}_2 = (-2)\cdot 2 + 1\cdot 4 + 0\cdot 5 = \underline{0}$

$\underline{v}_1 \cdot \underline{v}_3 = \underline{0}$

$\underline{v}_2 \cdot \underline{v}_2 = 4 + 16 + 25 = \underline{45}$

$\underline{v}_3 \cdot \underline{v}_2 = \underline{v}_2 \cdot \underline{v}_3 = 0$

$\underline{v}_2 \cdot \underline{v}_3 = \underline{0}$

$\underline{v}_3 \cdot \underline{v}_3 = \underline{9}$

$\underline{v}_3 \cdot \underline{v}_1 = \underline{v}_1 \cdot \underline{v}_3 = 0$

$\underline{v}_2 \cdot \underline{v}_1 = \underline{v}_1 \cdot \underline{v}_2 = 0$

P er <u>ikke</u> ortogonal.

Velg $\underline{v_1} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$.

$\underline{\lambda = 6:}$

$$\begin{pmatrix} -5 & 0 & 2 \\ 0 & -5 & 4 \\ 2 & 4 & -4 \end{pmatrix} \; 2/5 \; \rightarrow \begin{pmatrix} -5 & 0 & 2 \\ 0 & -5 & 4 \\ 0 & 4 & 4/5-4 \end{pmatrix} \; 4/5$$

$$\begin{pmatrix} -5 & 0 & 2 \\ 0 & -5 & 4 \\ 0 & 0 & 4/5-4+16/5 \end{pmatrix} = \begin{pmatrix} -5 & 0 & 2 \\ 0 & -5 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left.\begin{array}{r} -5x+2z=0 \\ -5y+4z=0 \\ z \text{ fri} \end{array}\right\} \quad \underline{x} = \begin{pmatrix} 2/5\, z \\ 4/5\, z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 2/5 \\ 4/5 \\ 1 \end{pmatrix} = \frac{z}{5} \cdot \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

Velg $\underline{v_2} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$.

$\underline{\lambda = -3:}$

$$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & 4 \\ 2 & 4 & 5 \end{pmatrix} \; -1/2 \; \rightarrow \begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & +4 \\ \cancel{0} & \cancel{4} & \cancel{4} \end{pmatrix} \qquad \begin{array}{r} 4x+2z=0 \\ 4y+4z=0 \\ z \text{ fri} \end{array}$$

$$\underline{x} = \begin{pmatrix} -z/2 \\ -z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1/2 \\ -1 \\ 1 \end{pmatrix} = \frac{z}{2} \cdot \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$$

Velg $\underline{v_3} = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$.

Velg $P = (\underline{v_1} | \underline{v_2} | \underline{v_3}) = \underline{\underline{\begin{pmatrix} -2 & 2 & -1 \\ 1 & 4 & -2 \\ 0 & 5 & 2 \end{pmatrix}}}$; da er

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

Forklaring:

$$P = (\underline{v}_1 \mid \underline{v}_2 \mid \underline{v}_3) \Rightarrow A \cdot P = A \cdot (\underline{v}_1 \mid \underline{v}_2 \mid \underline{v}_3)$$

$$= (A\underline{v}_1 \mid A\underline{v}_2 \mid A\underline{v}_3)$$

$$= (\lambda_1 \underline{v}_1 \mid \lambda_2 \underline{v}_2 \mid \lambda_3 \underline{v}_3)$$

$$= (\underline{v}_1 \mid \underline{v}_2 \mid \underline{v}_3) \cdot \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$= P \cdot D$$

$$AP = PD \Rightarrow P^{-1}AP = P^{-1}PD = D$$

$$\underline{P^{-1}AP = D}$$

d) $$f(x,y,z) = \overbrace{x^2 + 4xz + y^2 + 8yz + 2z^2}^{\text{kvadr.}} \underbrace{+6y - 12z}_{\text{lin.}} \underbrace{+8}_{\text{konst.}}$$

$$= \underbrace{\underline{x}^T A \underline{x}}_{\text{kvadr.}} + \underbrace{B\underline{x}}_{\text{lin.}} + \underbrace{C}_{\text{konst.}}, \quad \text{der}$$

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 2 & 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 6 & -12 \end{pmatrix}, \quad C = (8)$$

Stasjonære pkt:
$$\frac{\partial f}{\partial \underline{x}} = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2A \cdot \underline{x} + B^T = \underline{0}$$

$$A \cdot \underline{x} = -\frac{1}{2} B^T$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 2 & 4 & 2 \end{pmatrix} \underline{x} = \begin{pmatrix} 0 \\ -3 \\ 6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 4 & -3 \\ 2 & 4 & 2 & 6 \end{array}\right) \begin{matrix} \\ \\ \end{matrix} -2 \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 4 & -3 \\ 0 & 4 & -2 & 6 \end{array}\right) -4$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -18 & 18 \end{array}\right) \qquad \begin{aligned} x + 2z &= 0 \\ y + 4z &= -3 \\ -18z &= 18 \end{aligned} \qquad \begin{aligned} x &= 2 \\ y &= 1 \\ \underline{\underline{z}} &\underline{\underline{= -1}} \end{aligned}$$

Stasj. pkt: $\underline{\underline{(2, 1, -1)}}$

Punktet er et sadelpkt siden A er indefinitt.

Et stasjonært pkt for $f(\underline{x}) = \underline{x}^T A \underline{x} + b\underline{x} + c$ er

globalt maks. $\Longleftrightarrow$ A neg. semidefn.

globalt min $\Longleftrightarrow$ A pos. semidefn.

Sadelpkt $\Longleftrightarrow$ A indefinitt

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e) En kvadr. matrise $B = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n})$ er ortogonal hvis og bare hvis $B^{-1} = B^T$.

B ortogonal $\Longleftrightarrow$ $\underbrace{\begin{pmatrix} \underline{v_1} \\ \underline{v_2} \\ \vdots \\ \underline{v_n} \end{pmatrix}}_{B^T} \cdot \underbrace{\left(\underline{v_1} \middle| \underline{v_2} \middle| \dots \middle| \underline{v_n} \right)}_{B} = I$

$(B^T B = I$

$B \cdot B^T = I)$

$\Updownarrow$

$$\begin{pmatrix} \underline{v_1} \cdot \underline{v_1} & \underline{v_1} \cdot \underline{v_2} & \dots & \underline{v_1} \cdot \underline{v_n} \\ \vdots & \vdots & & \vdots \end{pmatrix} = I$$

$\underline{v_1} \cdot \underline{v_1} = 5$

$\underline{v_1}' \cdot \underline{v_1}' = \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \cdot 5 = 1$

Velg $\underline{v_1}' = \frac{1}{\sqrt{5}} \underline{v_1}$

$\underline{v_2} \cdot \underline{v_2} = 45$

$\underline{v_3} \cdot \underline{v_3} = 9$

Velg $\underline{v_2}' = \frac{1}{\sqrt{45}} \underline{v_2}$

Velg $\underline{v_3}' = \frac{1}{3} \underline{v_3}$

Vi kunne valgt $P = \begin{pmatrix} -2/\sqrt{5} & 2/\sqrt{45} & -1/3 \\ -1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ 0 & 5/\sqrt{45} & 2/3 \end{pmatrix}$

da er P ortogonal.

$\langle \underline{v}, \underline{w} \rangle$

Husk: $\underbrace{\underline{v} \cdot \underline{w}}_{\text{indreprodukt eller prikk-produkt}} = v_1 \cdot w_1 + v_2 \cdot w_2 + \dots + v_n \cdot w_n$

$= \underbrace{(v_1 \; v_2 \dots v_n) \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}}_{\text{matrise multiplikasjon}} = \underline{v}^T \underline{w}$

(f) $f(\underline{x}) = \underline{x}^t A \underline{x} + B\underline{x} + C$

$$\boxed{\begin{array}{l}\text{uten kryssledd} = \text{diagonalisert } A \\ P^{-1}AP = D \iff A = PDP^{-1}\end{array}}$$

$$\underline{x}^t A \underline{x} = \underline{x}^t P D P^{-1} \underline{x}$$ hvis P ortogonal

$$= \underline{x}^t P D P^t \underline{x}$$

$$= \underline{u}^t D \underline{u}$$

$\underline{u} = P^t \cdot \underline{x}$

$\underline{u}^t = \underline{x}^t P$

Derfor: Velg P ortogonal og $\underline{u} = P^t \cdot \underline{x}$

$$P = \begin{pmatrix} \frac{-2}{\sqrt{5}} & \frac{2}{\sqrt{45}} & -\frac{1}{3} \\ \frac{-1}{\sqrt{5}} & \frac{4}{\sqrt{45}} & -\frac{2}{3} \\ 0 & \frac{5}{\sqrt{45}} & \frac{2}{3} \end{pmatrix}, \quad \begin{pmatrix} u \\ v \\ w \end{pmatrix} = P^t \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 2/\sqrt{45} & 4/\sqrt{45} & 5/\sqrt{45} \\ -1/3 & -2/3 & 2/3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\boxed{\begin{array}{l} u = -\frac{2}{\sqrt{5}} x - \frac{1}{\sqrt{5}} y \\ v = \frac{2}{\sqrt{45}} x + \frac{4}{\sqrt{45}} y + \frac{5}{\sqrt{45}} z \\ w = -\frac{1}{3} x - \frac{2}{3} y + \frac{2}{3} z \end{array}}$$

Vi velger $\underline{u} = P^t \underline{x}$ med P ortogonal.

$\Updownarrow$

Da blir $\boxed{\underline{x} = P \cdot \underline{u}}$

$$f(\underline{x}) = \underline{x}^t A \underline{x} + B\underline{x} + C$$

$$= \underline{u}^t D \underline{u} + B \cdot P\underline{u} + C$$

$$= (u \ v \ w) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -3 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} +$$

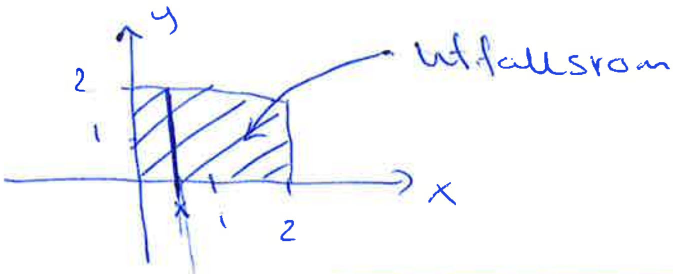
$$+ (0 \quad 6 \quad -12) \cdot \begin{pmatrix} -2/\sqrt{5} & 2/\sqrt{45} & -1/3 \\ -1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ 0 & 5/\sqrt{45} & 2/3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$+ 8$$

$$= u^2 + 6v^2 - 3w^2 - \frac{6}{\sqrt{5}} u - \frac{36}{\sqrt{45}} v - 12w + 8$$

Oppgave 2:

$$f(x,y) = k(x^3 + 3xy + y^3) \quad \text{når} \quad 0 \le x, y \le 2$$



a) $\boxed{\underline{\text{Sannsynlighetstetthet}} \quad \begin{array}{ll} \text{i)} & f(x,y) \ge 0 \text{ for alle } x,y. \checkmark \\ \text{ii)} & \iint f(x,y)\,dy\,dx = 1 \end{array}}$

$$f_x(x) = \int_0^2 k(x^3 + 3xy + y^3)\,dy = k\left[x^3 y + 3x \cdot \tfrac{1}{2}y^2 + \tfrac{1}{4}y^4\right]_0^2$$

$$= k(2x^3 + 6x + 4)$$

$$\int_0^2 k(2x^3 + 6x + 4)\,dx = k\left[2 \cdot \tfrac{1}{4}x^4 + 6 \cdot \tfrac{1}{2}x^2 + 4x\right]_0^2$$

$$= k \cdot (8 + 12 + 8) = 28k = 1 \Rightarrow \underline{\underline{k = \tfrac{1}{28}}}$$

$$\underline{\underline{f_x(x) = \tfrac{1}{28}(2x^3 + 6x + 4)}}$$

b) Finn $E(X)$, $Var(X) = E(X^2) - E(X)^2$

$\boxed{\underline{\text{Husk:}} \text{ Bruk } f_x(x) \text{ til å finne } E(X), E(X^2), \ldots}$

$$E(X) = \int_0^2 x \cdot \frac{1}{28}(2x^3 + 6x + 4)\, dx$$

$$= \frac{1}{28} \int_0^2 2x^4 + 6x^2 + 4x \, dx$$

$$= \frac{1}{28} \left[2 \cdot \frac{1}{5}x^5 + 6 \cdot \frac{1}{3}x^3 + 4 \cdot \frac{1}{2}x^2 \right]_0^2$$

$$= \frac{1}{28}\left(\frac{2}{5} \cdot 32 + 2 \cdot 8 + 2 \cdot 4 \right)$$

$$= \frac{1}{28}\left(\frac{64}{5} + 16 + 8 \right) = \frac{1}{28} \cdot \frac{64 + 5 \cdot 24}{5}$$

$$= \frac{184}{28 \cdot 5} = \frac{92}{14 \cdot 5} = \frac{46}{7 \cdot 5} = \underline{\underline{\frac{46}{35}}}$$

$$E(X^2) = \int_0^2 x^2 \cdot \frac{1}{28}(2x^3 + 6x + 4)\, dx$$

$$= \frac{1}{28} \left[2 \cdot \frac{1}{6}x^6 + 6 \cdot \frac{1}{4}x^4 + 4 \cdot \frac{1}{3}x^3 \right]_0^2$$

$$= \frac{1}{28}\left(\frac{1}{3} \cdot 64 + \frac{6}{4} \cdot 16 + \frac{4}{3} \cdot 8 \right)$$

$$= \frac{1}{28}\left(\frac{64}{3} + \frac{48}{2} + \frac{32}{3} \right) = \frac{1}{28}(32 + 24)$$

$$= \underline{2} \Rightarrow Var(X) = \underline{\underline{2 - \left(\frac{46}{35}\right)^2}} \leftarrow$$

forkort eller desimaltall

c) $\text{Cov}(X,Y) = E(XY) - E(X)\cdot E(Y)$

Vet at $E(X) = \frac{46}{35}$, og $E(Y) = E(X) = \frac{46}{35}$

fordi $f(x,y) = f(y,x)$

$$E(XY) = \int_0^2\int_0^2 xy \cdot \frac{1}{28}(x^3 + 3xy + y^3)\, dy\, dx$$

$$= \frac{1}{28}\int_0^2 \left(\int_0^2 x^4 y + 3x^2y^2 + xy^4\, dy \right) dx$$

$$= \frac{1}{28}\int_0^2 \left[x^4 \cdot \frac{1}{2}y^2 + 3x^2 \cdot \frac{1}{3}y^3 + x \cdot \frac{1}{5}y^5 \right]_0^2 dx$$

$$= \frac{1}{28}\int_0^2 2x^4 + 8x^2 + \frac{32}{5}x \qquad dx$$

$$= \frac{1}{28}\left[2\cdot\frac{1}{5}x^5 + 8\cdot\frac{1}{3}x^3 + \frac{32}{5}\cdot\frac{1}{2}x^2 \right]_0^2$$

$$= \frac{1}{28}\left(\frac{64}{5} + \frac{64}{3} + \frac{64}{5} \right) = \frac{1}{28}\left(\frac{128\cdot 3}{5\cdot 3} + \frac{64\cdot 5}{3\cdot 5} \right)$$

$$= \frac{384 + 320}{28\cdot 5\cdot 3} = \underline{\underline{\frac{704}{28\cdot 5\cdot 3}}} \quad \leftarrow$$ forkort eller desimaltall

$$\text{Cov}(X,Y) = \underline{\underline{\frac{704}{28\cdot 15} - \left(\frac{46}{35}\right)^2}} \quad \leftarrow$$ forkort eller desimaltall

BI

Er X og Y uavhengige ?

$$\boxed{\begin{array}{ccc} X \text{ og } Y \text{ uavhengige} & \Longrightarrow & \mathrm{Cov}(X,Y)=0 \\ \Updownarrow & & \\ f(x,y) = f_X(x)\cdot f_Y(y) & & \end{array}}$$

Siden $\mathrm{Cov}(X,Y) \neq 0$ så er X og Y <u>ikke</u> uavhengige.

d) $Z = 2U + 3V$, <u>U og V uavh</u>

$\Downarrow$

$E(Z) = 2E(U) + 3E(V) = 2\cdot 1 + 3\cdot 3 = \underline{\underline{11}}$

$\mathrm{Var}(Z) = \mathrm{Cov}(Z,Z) = \mathrm{Cov}(2U+3V, 2U+3V)$

$= 2^2\cdot \mathrm{Cov}(U,U) + 2\cdot 3\cdot \mathrm{Cov}(U,V) + 3\cdot 2\cdot \mathrm{Cov}(V,U) + 3\cdot 3\cdot \mathrm{Cov}(V,V)$

$= 4\,\mathrm{Var}(U) + 9\,\mathrm{Var}(V) + 12\,\mathrm{Cov}(U,V)$

$= 4\cdot 4 + 9\cdot 1 = \underline{\underline{25}}$

e) Var(Z) skal uttrykkes vha $c = E(UV)$:

$\mathrm{Var}(Z) = 25 + 12\cdot\left(E(UV) - E(U)\cdot E(V)\right)$

$= 25 + 12\,(c - 3) = \underline{\underline{12c - 11}}$

U og V avh. $\Longrightarrow$ $\mathrm{Cov}(U,V) = 0$

$c - 3 = 0$

$\underline{\underline{c = 3}}$

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Oppgave 3

Diff. likn.:

i) $y'' + ay' + by = f(t)$
ii) $y' + a(t)y = b(t)$

} lineære diff. likn.

$y = y_h + y_p$ karakterisk likn., eller int. faktor (ii) når $a(t)$ ikke er konstant)

iii) $y' = f(y) \cdot g(t)$ } Seperabel

a) $y'' - 8y' - 7y = 21$

$$y = y_h + y_p = \underline{\underline{C_1 e^{(4+\sqrt{23})t} + C_2 e^{(4-\sqrt{23})t} - 3}}$$

y_h: $r^2 - 8r - 7 = 0$

~~r = 4, 7~~

$$r = \frac{8 \pm \sqrt{64 + 28}}{2}$$

$$= \frac{8 \pm \sqrt{92}}{2} = \underline{4 \pm \sqrt{23}}$$

$$y_h = C_1 \cdot e^{(4+\sqrt{23})t} + C_2 \cdot e^{(4-\sqrt{23})t}$$

y_p: $y_p = -3$

b) $t^2 y' - y = 1$ lineær, $: t^2$

$$y' - \frac{1}{t^2} y = \frac{1}{t^2}$$

$$y' + a(t)y = b(t)$$

$$a(t) = -\frac{1}{t^2} \qquad b(t) = \frac{1}{t^2}$$

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<u>Integrerende faktor:</u>

$$a(t) = -\frac{1}{t^2}$$

$$\int a(t)\,dt = \int -\frac{1}{t^2}\,dt$$

$$= \int -t^{-2}\,dt = \cancel{-1}\cdot\frac{t^{-2+1}}{\cancel{-2+1}} + C$$

$$= t^{-1} + C$$

$$\Downarrow$$

Int. faktor : $e^{t^{-1}} = e^{1/t}$

$$y' - \frac{1}{t^2}y = \frac{1}{t^2} \quad \Big|\cdot e^{1/t}$$

$$(y\cdot e^{1/t})' = \frac{1}{t^2}\cdot e^{1/t}$$

$$y\cdot e^{1/t} = \int \frac{1}{t^2}e^{1/t}\,dt$$

$$= \int e^u\cdot(-1)\,du = -e^u + C$$

<u>substitusjon:</u>

$u = 1/t$

$u' = -t^{-2} = -\frac{1}{t^2}$

$du = -\frac{1}{t^2}\,dt$

$$y\cdot e^{1/t} = -e^{1/t} + C$$

$$\underline{\underline{y = -1 + C\,e^{-1/t}}}$$

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c) $e^{-y} + ty' = 1$

$$ty' = 1 - e^{-y}$$

$$y' = \frac{1}{t} \cdot (1 - e^{-y}) \Leftarrow \text{separabel}$$

$$\frac{1}{1-e^{-y}} y' = \frac{1}{t}$$

$$\boxed{\int \frac{1}{1-e^{-y}} dy = \int \frac{1}{t} dt}$$

$$\int \frac{1}{1 - \frac{1}{e^y}} dy = \ln t + C$$

$$\int \frac{e^y}{e^y - 1} dy = \ln t + C$$

Substitusjon:

$u = e^y - 1$

$du = e^y \cdot dy$

$$\int \frac{1}{u} du = \ln u + C = \ln(e^y - 1) + C$$

Dermed:

$$\ln |e^y - 1| = \ln|t| + C$$

$$|e^y - 1| = e^{\ln t + C} = |t| e^C$$

$$e^y - 1 = t \cdot K \qquad (K = \pm e^C)$$

$$e^y = Kt + 1$$

$$\underline{\underline{y = \ln(Kt+1)}}$$

Oppgave 4

$$\max \int_0^4 \ln(y-y')\, e^{-rt}\, dt \quad \text{når} \quad \begin{cases} y(0)=1 \\ y(4)=e^{4-4\ldots} \end{cases}$$

a) $F = \ln(y-y')\, e^{-rt}$

$$F'_y = \frac{1}{y-y'} e^{-rt}$$

Euler:

$$F'_y - \frac{d}{dt}\left(F'_{y'}\right) = 0$$

$$F'_{y'} = \frac{-1}{y-y'} \cdot e^{-rt}$$

$$\frac{d}{dt}\left(\frac{-1}{y-y'} e^{-rt}\right) = (-1)(y-y')^{-2} \cdot (y'-y'') \cdot (-1) \cdot e^{-rt}$$

$$+ \frac{-1}{y-y'} \cdot e^{-rt} \cdot (-r)$$

$$= e^{-rt}\left(\frac{y'-y''}{(y-y')^2} + \frac{r}{y-y'}\right)$$

Euler: $\frac{1}{y-y'} e^{-rt} - e^{-rt}\left(\frac{y'-y''}{(y-y')^2} + \frac{r}{y-y'}\right) = 0$

$\cdot\, e^{rt}$ gir $\quad \frac{1}{y-y'} - \frac{y'-y''}{(y-y')^2} - \frac{r}{y-y'} = 0$

$\cdot\, (y-y')^2$ gir:

$$y - y' - (y'-y'') - r(y-y') = 0$$

$$y'' + (r-2)y' + (1-r)y = 0$$

Kar. likn. (med ukjent λ): $\quad \lambda^2 + (r-2)\lambda + (1-r) = 0$

$$\lambda = \frac{2-r \pm \sqrt{(r-2)^2 - 4\cdot(1-r)}}{2}$$

$$= \frac{2-r \pm \sqrt{r^2 \cancel{-4r} \cancel{+4} \cancel{-4} \cancel{+4r}}}{2}$$

$$= \frac{2-r \pm r}{2} = \underline{1}, \quad \frac{2-2r}{2} = \underline{1-r}$$

$$\underline{y^* = C_1 e^{t} + C_2 e^{(1-r)t}}$$ (løsn. av Euler)

$y(0) = 1$

$y(4) = e^{4-4r}$

$C_1 + C_2 = 1$

$C_1 e^{4} + C_2 e^{4-4r} = e^{4-4r}$

$\underline{\underline{C_1 = 0,\ C_2 = 1}}$ (løs dette ved å løse likn.)

$$y^* = \underline{\underline{e^{(1-r)t}}}$$

b) $\boxed{\text{Husk: } F \text{ konkav i } (y, y') \Rightarrow y^* \text{ gir maks}}$

$$F'_y = \frac{1}{y-y'} e^{-rt}$$

$$F'_{y'} = \frac{-1}{y-y'} e^{-rt}$$

$$F''_{yy} = -1 \cdot \frac{1}{(y-y')^2} \cdot 1 \cdot e^{-rt}$$

$$= \frac{-1}{(y-y')^2} e^{-rt} < 0$$

$$F''_{yy'} = \frac{1}{(y-y')^2} e^{-rt}$$

$$F''_{y'y'} = (-1)\cdot(-1)\cdot(y-y')^{-2}\cdot(-1)\, e^{-rt}$$

$$= -\frac{1}{(y-y')^2} e^{-rt} < 0$$

Hessisk matrise:

$$H(F) = \frac{e^{-rt}}{(y-y')^2} \cdot \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\det H(F) = \left(\frac{e^{-rt}}{(y-y')^2}\right)^2 \cdot \left((-1)^2 - 1^2 \right) = 0$$

Derfor er F konkav i (y, y')

$$\Downarrow$$

$$\underline{\underline{y^* = e^{(1-r)t} \text{ gir maks}}}$$