Forelesning 8 ELE 3719

Eivind Eriksen Mar 03, 2015 Matematikk

Plan:

① Kontinuerlige stokastiske variable

② Chebyshov's ulikhet

Pensum:

[S] 5.4 - 5.9

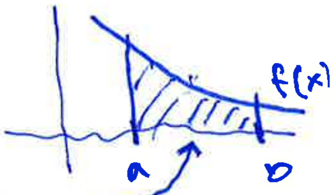
① Kontinuerlige stokastiske variable

En stokastisk variabel X, der de mulige verdiene av X er en <u>kontinuerlig mengde</u> kalles en kontinuerlig stokastisk variabel.

Eks:
Vekten av en tilfeldig valgt laks
Mulige verdier: $[0, 30]$

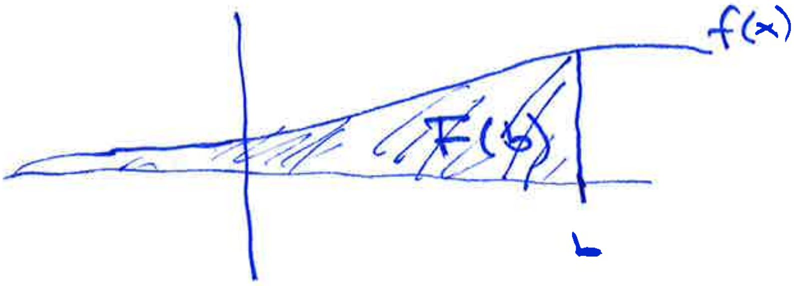
I så fall fins det en kontinuerlig funksjon $f(x) = f_X(x)$ som kalles <u>sannsynlighetstettheten</u> til X, slik at

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

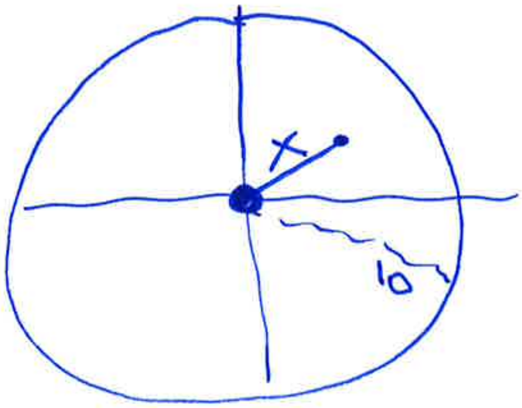


Fordelingsfunksjon til X er

$$F(b) := p(X \le b) = \int_{-\infty}^{b} f(x)\,dx$$



Eks:



X = avstand til senteret

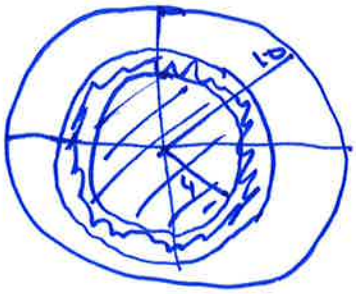
Mulige verdiene for X: $[0,10]$

$\Downarrow$

X kontinuerlig stokastisk variabel

Vi antar at alle punkter innenfor den store sirkelen er like sannsynlige.

$$p(X \le 4) = \frac{\cancel{\pi} \cdot 4^2}{\cancel{\pi} \cdot 10^2} = \frac{16}{100} = 0.16$$



$$p(X \le 5) = \frac{\cancel{\pi} \cdot 5^2}{\cancel{\pi} \cdot 10^2} = \frac{25}{100} = 0.25$$

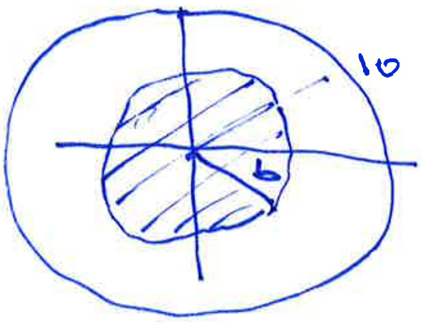
$$p(4 \le X \le 5) = 0.25 - 0.16 = 0.09$$

$$p(X = 4) = 0$$

Punktsannsynlighetene er alltid null for en kont. variabel.

Hva blir $f(x)$ og $F(x)$ i eks.?

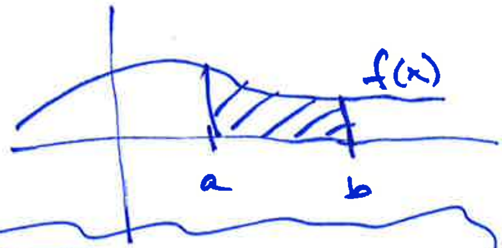
$$F(b) = P(X \le b) = \frac{\cancel{\pi} \cdot b^2}{\cancel{\pi} \cdot 10^2} = \frac{b^2}{100}$$



$$F(x) = \begin{cases} \dfrac{x^2}{100}, & 0 \le x \le 10 \end{cases}$$

Hva er $f(x)$?

$$\int_a^b f(x)\,dx = P(a \le X \le b)$$



$$\boxed{\int_{-\infty}^{b} f(x)\,dx = F(b)}$$

$$\Downarrow$$

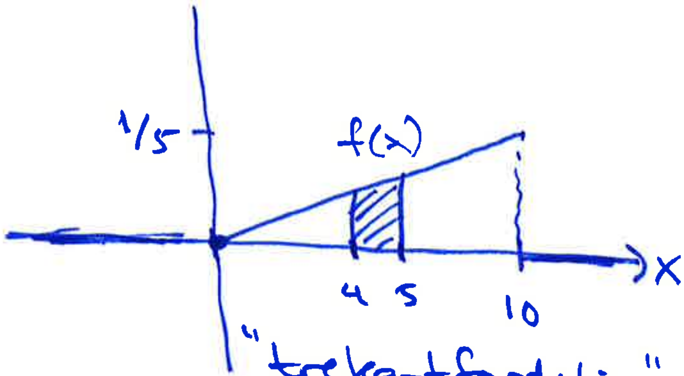
Dette gjelder for alle kontinuerlige variable:

$$\boxed{f(x) = F'(x)}$$

I eks: $F(x) = \dfrac{x^2}{100}, \quad 0 \le x \le 10$

$$\Downarrow$$

$$f(x) = \frac{2x}{100} = \underline{\underline{\frac{x}{50}}}, \quad 0 \le x \le 10$$



"trekantfordeling"

$$f(x) = \begin{cases} 0 & , x < 0 \\ \frac{x}{50} & , 0 \le x \le 10 \\ 0 & , x > 10 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{10} f(x)\,dx = 1$$

$$p(4 \le X \le 5) = \int_4^5 \frac{x}{50}\,dx = \frac{1}{50} \cdot \left[\frac{1}{2}x^2\right]_4^5$$

$$= \frac{1}{50}\left(\frac{1}{2}\cdot 5^2 - \frac{1}{2}\cdot 4^2\right)$$

$$= \frac{1}{100}\cdot(25-16) = \frac{9}{100} = \underline{\underline{0.09}}$$

X: kont. stokastisk variabel

$f(x)$: sanns. tetthetsfunksjon *

$F(x)$: kum. fordelingsfunksjon for X

Husk:

$$\boxed{\begin{aligned} &F(b) = p(X \le b) \\ &p(a \le X \le b) = \int_a^b f(x)\,dx \\ &F'(x) = f(x) \end{aligned}}$$

$p(X = a) = 0$
$\neq$
$f(a)$

Krav til $f(x)$:

i) $f(x) \ge 0$ for alle x

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Krav til $F(x)$:

i) $F(x)$ monotont voksende

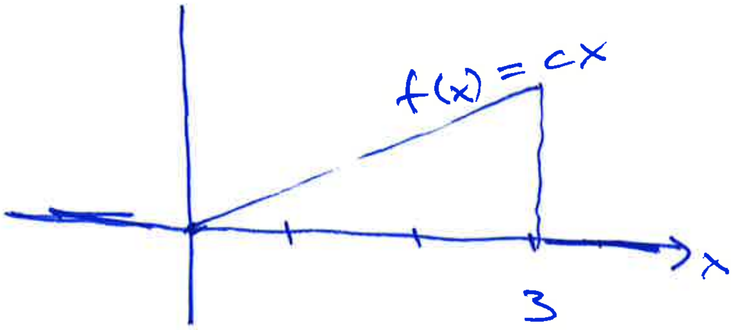
ii) $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$

Eks:

X er en kont. stokastisk variabel med tetthet

$$f(x) = cx, \quad 0 \le x \le 3$$

for et tall c. Underforstått $f(x)=0$ når $x<0$ og $x>3$.



$$\text{Areal} = \frac{3 \cdot 3c}{2} = \frac{9}{2}c = 1$$

$$c = \underline{\underline{2/9}}$$

Krav:

i) $f(x) = cx \ge 0$ for alle x : ok hvis $c \ge 0$

ii) $\int_0^3 f(x)\,dx = 1$: $\int_0^3 cx\,dx = c\left[\frac{1}{2}x^2\right]_0^3$

$$= c \cdot \frac{1}{2} \cdot (3^2 - 0^2) = \frac{9}{2}c = 1$$

$$c = \underline{\underline{\frac{2}{9}}}$$

Konklusjon.

$f(x) = \frac{2}{9}x, \quad 0 \le x \le 3$ er en tetthetsfunksjon til en kont. stokastisk variabel.

Forventning og varians:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

Eks: $X: \quad f(x) = \frac{x}{50}, \quad 0 \le x \le 10$

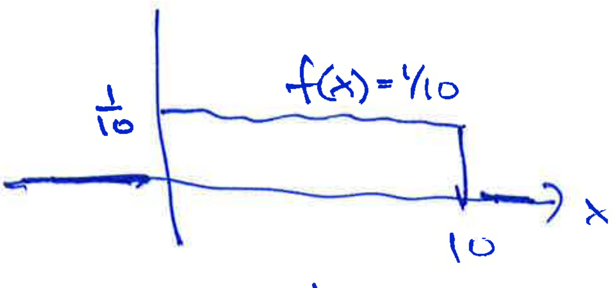
$$E(X) = \int_0^{10} x \cdot \frac{x}{50}\, dx = \frac{1}{50} \cdot \left[\frac{1}{3}x^3\right]_0^{10}$$

$$= \frac{1}{150} \cdot 10^3 = \frac{1000}{150} = \frac{100}{15} = \underline{\underline{\frac{20}{3}}}$$

$$\simeq 6.67$$

Eks: $X: \quad f(x) = \frac{1}{10}, \quad 0 \le x \le 10$

(uniform fordeling)



$$E(X) = \int_0^{10} x \cdot \frac{1}{10}\, dx = \frac{1}{10} \cdot \left[\frac{1}{2}x^2\right]_0^{10}$$

$$= \frac{1}{20} \cdot 100 = \underline{\underline{5}}$$

$$\mathrm{Var}(X) = E\left((X-\mu)^2\right)$$ når $\mu = E(X)$ er endelig

Vi regner ut $E(h(X))$ ved hjelp av:

$$E[h(X)] = \int_{-\infty}^{\infty} h(x)\cdot f(x)\,dx$$

<u>Eks:</u> $f(x) = \frac{1}{10}$, $0 \leq x \leq 10$

$E(X) = 5$ $\quad \mu = 5$

$$\mathrm{Var}(X) = E\left[(X-5)^2\right]$$

$$= \int_0^{10} (x-5)^2 \cdot \frac{1}{10}\,dx$$

$$= \frac{1}{10}\left[\frac{1}{3}(x-5)^3\right]_0^{10}$$

$$= \frac{1}{30}\left(5^3 - (-5)^3\right)$$

$$= \frac{1}{30}(125+125) = \frac{250}{30} = \frac{25}{3}$$

$$\simeq \underline{\underline{8.33}}$$

$$\boxed{\text{Husk: } E[h(X)] = \int_{-\infty}^{\infty} h(x)\cdot f(x)\,dx}$$

Regneregler for forventning og varians

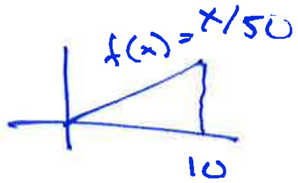
X : kont. stok. variabel
a, b : tall

i) $E(aX+b) = a \cdot E(X) + b$

ii) $\mathrm{var}(X) = E(X^2) - E(X)^2$

iii) $\mathrm{var}(aX+b) = a^2 \cdot \mathrm{var}(X)$

Eks: $f(x) = \frac{x}{50}, \quad 0 \le x \le 10$



$E(X) = \frac{20}{3} \simeq 6{,}67$

$\mathrm{Var}(X) = E\left[(X - 6.67)^2\right] = \dots$

$= E(X^2) - E(X)^2 = 50 - \left(\frac{20}{3}\right)^2$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^{10} x^2 \cdot \frac{x}{50}\,dx$$

$$= \frac{1}{50}\left[\frac{1}{4}x^4\right]_0^{10} = \frac{1}{200}\left(10^4\right)$$

$$= \frac{10.000}{200} = 50$$

$$\mathrm{Var}(X) = 50 - \left(\frac{20}{3}\right)^2 = 50 - \frac{400}{9}$$

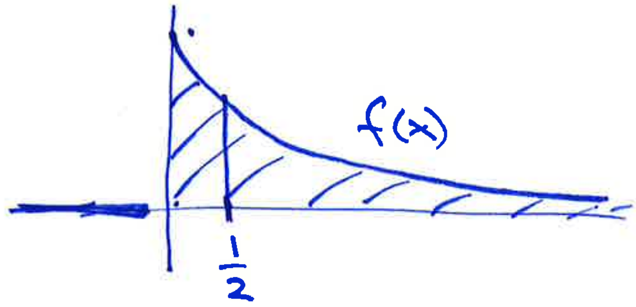
$$= \frac{450 - 400}{9} = \underline{\underline{\frac{50}{9}}}$$

Eksponentiell fordeling: Parameter λ

$$f(x) = \lambda \cdot e^{-\lambda x} \quad , x \geq 0$$

Eks: $\lambda = 2$ $\qquad f(x) = 2e^{-2x} \quad , x \geq 0$

Sjekke at dette er en fordeling:



$f(x) \geq 0$ ok.

$\int_0^\infty f(x)\,dx = 1$ ok.

$$\int_0^\infty 2e^{-2x}\,dx = \left[-e^{-2x}\right]_0^\infty$$

$$= \lim_{b\to\infty} \left[-e^{-2x}\right]_0^b = \lim_{b\to\infty}\left(-e^{-2b} + 1\right)$$

$$= \lim_{b\to\infty}\left(1 - \frac{1}{e^{2b}}\right) = 1$$

$$E(x) = \int_0^{\infty} x \cdot \underbrace{2e^{-2x}}_{f(x)} \, dx = \lim_{b \to \infty} \int_0^{b} x \cdot 2e^{-2x} \, dx$$

$$\int u'v \, dx = uv - \int uv' \, dx$$

delvis int.

$$\int_0^b x \cdot 2e^{-2x} \, dx = \int_0^b u' v \, dx$$

$$u' = 2e^{-2x} \qquad v = x$$

$$u = -e^{-2x} \qquad v' = 1$$

$$= \left[-e^{-2x} \cdot x \right]_0^b - \int_0^b -e^{-2x} \cdot 1 \, dx$$

$$= -be^{-2b} + \int_0^b e^{-2x} \, dx$$

$$= -be^{-2b} + \left[\frac{-1}{2} \cdot e^{-2x} \right]_0^b$$

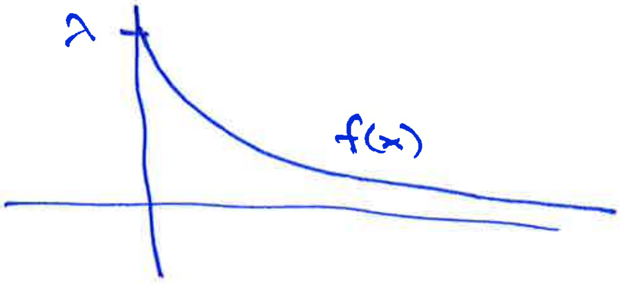
$$= -be^{-2b} - \frac{1}{2} e^{-2b} + \frac{1}{2}$$

$$E(x) = \lim_{b \to \infty} \left(-be^{-2b} - \frac{1}{2} e^{-2b} + \frac{1}{2} \right)$$

$$= \frac{1}{2} - \underbrace{\lim_{b \to \infty} \frac{b}{e^{2b}}}_{=0} - \underbrace{\lim_{b \to \infty} \frac{1}{2e^{2b}}}_{=0} = \underline{\underline{1/2}}$$

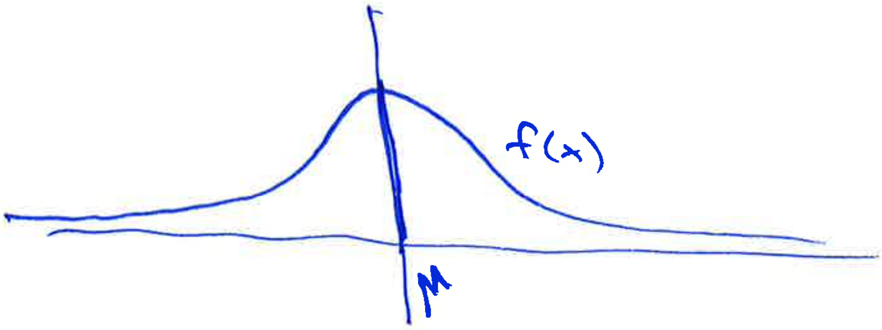
i) <u>Eksponentiell fordeling:</u>

$f(x) = \lambda \cdot e^{-\lambda x}$, $x \geq 0$ $E(x) = \frac{1}{\lambda}$ $var(x) = \frac{1}{\lambda^2}$



ii) <u>Normalfordeling:</u>

$$f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad , \text{ for alle } x$$



Grafen når $\mu = 0, \sigma = 1$

$E(x) = \mu$ $Var(x) = \sigma^2$

<u>$\mu = 0, \sigma = 1$:</u>

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

(2) Noen viktige ulikheter:

Chebyshev's ulikhet:

$$\boxed{P\left(\left|\frac{X-\mu}{\sigma}\right| > k\right) \leq \frac{1}{k^2}}$$

For enhver stokastisk variabel X med $E(X)=\mu$, $Var(X)=\sigma^2$, og ethvert positivt tall k.

Hvis X har forventning μ og std. avvik σ

$$E(X)=\mu \quad , \quad Var(X)=\sigma^2$$

så kan vi se på

$$Z = \frac{X-\mu}{\sigma} = \frac{1}{\sigma}\cdot(X-\mu) = \frac{1}{\sigma}\cdot X - \frac{\mu}{\sigma}$$

Da er $E(Z) = \frac{1}{\sigma}E(X) - \frac{\mu}{\sigma} = \frac{\mu}{\sigma} - \frac{\mu}{\sigma} = \underline{0}$

$Var(Z) = \frac{1}{\sigma^2}\cdot Var(X) = \frac{1}{\sigma^2}\cdot\sigma^2 = \underline{1}$

Tolkning:

$P\left(\left|\frac{X-\mu}{\sigma}\right| > k\right)$ = sannsynlighet for at X er minst k std. avvik unna forventningen.

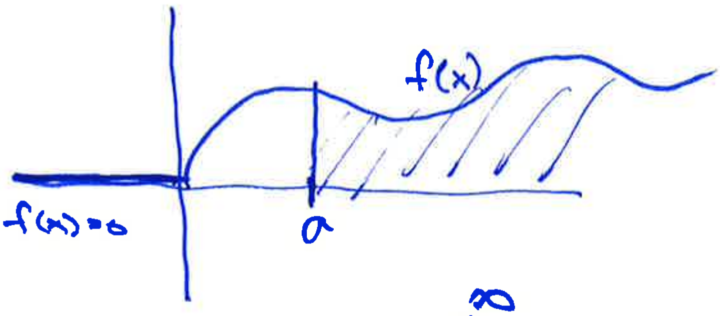
$\underline{k=2}$: $\quad P\left(\left|\frac{X-\mu}{\sigma}\right| > \underset{\underset{2}{\shortparallel}}{k}\right)$ = Sannsynlighet for at X er minst to std avvik unna μ

C's ulikhet: $\leq 1/4$

Bevis:

Markov's ulikhet:

Hvis $X \geq 0$, så er $\quad P(X \geq a) \leq \frac{E(X)}{a}$



$$E(x) = \int_0^\infty x \cdot f(x)\,dx = \int_0^a x \cdot f(x)\,dx + \int_a^\infty x f(x)\,dx$$

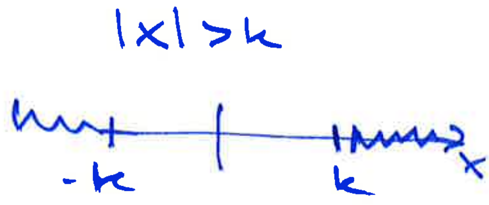
$$\geq \int_a^\infty x\, f(x)\,dx \geq \int_a^\infty a\, f(x)\,dx$$

$$= a \cdot \int_a^\infty f(x)\,dx = a \cdot P(X \geq a)$$

$$E(x) \geq a \cdot P(X \geq a) \quad \cdot \frac{1}{a}$$

$$\boxed{P(X \geq a) \leq \frac{E(X)}{a}}$$

Chebyshev's ulikhet:



$$\boxed{\begin{array}{l} \underline{x \geq 0:} \\ P(X \geq a) \\ \quad \leq \dfrac{E(X)}{a} \end{array}}$$

$$P\left(\left| \frac{X-\mu}{\sigma} \right| \geq k \right) =$$

$$P\left(\underbrace{\left(\frac{X-\mu}{\sigma} \right)^2}_{\geq 0} \geq k^2 \right)$$

$$\leq \frac{E\left[\left(\frac{X-\mu}{\sigma} \right)^2 \right]}{k^2} = \frac{E\left[\frac{1}{\sigma^2} (X-\mu)^2 \right]}{k^2}$$

$$= \frac{\frac{1}{\sigma^2} E\left((X-\mu)^2 \right)}{k^2} = \frac{\frac{1}{\sigma^2} \cdot Var(X)}{k^2}$$

$$= \frac{\frac{1}{\sigma^2} \cdot \sigma^2}{k^2} = \frac{1}{k^2}$$