FORELESNING 9

ELE 3719

EIVIND ERIKSEN   MAR 10, 2015   MATEMATIKK

Plan:

Pensum:

[S] 6.1 - 6.8

1. Simultant fordelte stokastiske variable
   a) Diskret tilfelle
   b) Kontinuerlig tilfelle
2. Dobbelt-integral
3. Kovarians og uavhengighet

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1. Simultant fordelte stokastiske variable

To stokastiske variable X, Y er simultant fordelte om begge avhenger av utfallet ω i det samme stokastiske forsøket.

$\Omega = \{\omega_i\}$ felles utfallsrommet

$X = X(\omega)$

$Y = Y(\omega)$

To tilfeller:

i) X, Y diskret

ii) X, Y kontinuerlig

a) <u>Diskret tilfelle:</u> $\left.\begin{array}{l} X = X(\omega) \\ Y = Y(\omega) \end{array}\right\}$ diskret variable

<u>Eks:</u> I : Vi kaster en rød og en blå terning

X = ant. øyne rød terning
Y = — " — blå terning

Y \ X	1	2	3	4	5	6	
1	1/36	1/36	1/36	1/36	1/36	1/36	$\frac{1}{6}$
2	1/36	1/36	1/36	. . .	.	.	$\frac{1}{6}$
3	.	.					
4							
5							
6							
	1/6	1/6					

$p(X=2, Y=1)$
$= p(X=2 \text{ og } Y=1)$
$= p(2,1) = \frac{1}{36}$

$p(x=2, y=1)$
$= p(2,1) = \frac{1}{36}$
$\shortparallel$
$p_X(2) \cdot p_Y(1)$
$= \frac{1}{6} \cdot \frac{1}{6}$
$\shortparallel$
X, Y uavhengige

<u>Simultan sannsynlighetsfordeling:</u>

$$p(x,y) = p(X=x, Y=y) = \frac{1}{36} \quad \text{for alle } x, y$$

<u>Kumulativ fordelingsfunksjon:</u>

$$F(a,b) = p(X \leq a, Y \leq b) = \sum_{\substack{x \leq a \\ y \leq b}} p(x,y)$$

$$F(3,2) = \sum_{\substack{x \leq 3 \\ y \leq 2}} p(x,y) = \frac{6}{36} = \frac{1}{6}$$

Eks 2:

$Y \backslash X$	1	2	3	
1	0.20	a	0.20	0.4+a
2	0.20	a	0.10	0.3+a
	0.40	2a	0.30	1

$p(x,y)$: $1 \leq x \leq 3$, $1 \leq y \leq 2$

$a = 0.15$

Krav til sannsynlighetsfunksjon:

i) $p(x,y) \geq 0$ for alle x,y

ii) $\sum_{x,y} p(x,y) = 1$

Marginale sannsynlighetstettheter:

$$P_X(1) = \sum_y p(1,y) = p(1,1) + p(1,2) = 0.40$$

$$P_X(2) = \sum_y p(2,y) = p(2,1) + p(2,2) = 2a$$

$$P_X(3) = \sum_y p(3,y) = p(3,1) + p(3,2) = 0.30$$

$$P_Y(1) = \sum_x p(x,1) = p(1,1) + p(2,1) + p(3,1) = 0.4 + a$$

$$P_Y(2) = \sum_x p(x,2) = p(1,2) + p(2,2) + p(3,2) = 0.3 + a$$

Vi sier at X og Y er uavhengige hvis

$$p(X=x, Y=y) = p(X=x) \cdot p(Y=y)$$

$$\boxed{p(x,y) = p_X(x) \cdot p_Y(y)}$$

Husk: Hendelsene A og B er uavhengige hvis

$$p(A \cap B) = p(A) \cdot p(B)$$

Eks 2:

Y \ X	1	2	3	
1	0.20	0.15	0.20	0.55
2	0.20	0.15	0.10	0.45
	0.40	0.30	0.30	1

$$\left.\begin{array}{l} p(2,1) = 0.15 \\ p_X(2) \cdot p_Y(1) = 0.30 \cdot 0.55 = 0.165 \end{array}\right\}$$ X og Y ikke uavhengige

Forventning, varians, kovarians

$$E(X) = \sum_x x \cdot p_X(x) = 1 \cdot 0.40 + 2 \cdot 0.3 + 3 \cdot 0.30 = \underline{1.9}$$

$$= \sum_{x,y} x \cdot p(x,y) = 1 \cdot 0.20 + 2 \cdot 0.15 + 3 \cdot 0.20 + 1 \cdot 0.20 + 2 \cdot 0.15 + 3 \cdot 0.10 = \underline{1.9}$$

$$E(X^2) = \sum_x x^2 \cdot p_X(x) = 1^2 \cdot 0.4 + 2^2 \cdot 0.3 + 3^2 \cdot 0.3 = \underline{4.3}$$

$$E(Y) = \sum_y y \cdot p_Y(y) = 1 \cdot 0.55 + 2 \cdot 0.45 = \underline{1.45}$$

$$E(Y^2) = \sum_y y^2 \cdot p_Y(y) = 1^2 \cdot 0.55 + 2^2 \cdot 0.45 = \underline{2.35}$$

$$E(XY) = \sum_{x,y} xy \cdot p(x,y) = 1 \cdot 0.20 + 2 \cdot 0.15 + 3 \cdot 0.20 + 2 \cdot 0.20 + 4 \cdot 0.15 + 6 \cdot 0.10$$

$$= \underline{2.7}$$

Generelt: $$E[h(X,Y)] = \sum_{x,y} h(x,y) \cdot p(x,y)$$

Som før:

$$E(X) = 1.9$$

$$Var(X) = E(X^2) - E(X)^2 = 4.3 - 1.9^2 = \ldots$$

$$E(Y) = 1.45$$

$$Var(Y) = E(Y^2) - E(Y)^2 = 2.35 - 1.45^2 = \ldots$$

Kovarians: $$Cov(X,Y) = E[(X - \mu_X) \cdot (Y - \mu_Y)]$$

der $\mu_X = E(X)$, $\mu_Y = E(Y)$

Merk:

$$Cov(X,Y) = E[XY - \mu_X \cdot Y - X \cdot \mu_Y + \mu_X \mu_Y]$$

$$= E(XY) - \mu_X (E(Y)) - \mu_Y \cdot E(X) + \mu_X \mu_Y$$

$$= E(XY) - \mu_X \mu_Y - \cancel{\mu_Y \mu_X} + \cancel{\mu_X \mu_Y}$$

$$\boxed{Cov(X,Y) = E(XY) - E(X) \cdot E(Y)}$$

Regneregler:

i) $E(aX+bY+c) = a \cdot E(X) + b \cdot E(Y) + c$

ii) $Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$

iii) $Cov(X,X) = Var(X)$

iv) $Cov(X,Y) = cov(Y,X)$

v) $cov(aX+bX', Y) = a \cdot Cov(X,Y) + b \cdot Cov(X',Y)$

Eks:

$$Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$$
$$= 2.7 - \underbrace{1.9 \cdot 1.45}_{} = \underline{\underline{-0.055}}$$

$2.90 - 0.145$

$= 2.755$

Korrelasjonskoeffisient: $\rho_{XY} = \dfrac{Cov(X,Y)}{\sqrt{VarX} \cdot \sqrt{Var(Y)}} = \dfrac{-0.055}{\cdot \quad \cdot}$

Viktig spesialtilfelle:

Hvis X og Y er uavhengige, da har vi:

i) $Cov(X,Y) = 0$

ii) $E[f(X) \cdot g(Y)] = E[f(X)] \cdot E[g(Y)]$

Begrunnelse: i) er et spesialtilfelle av ii) med $f(X) = X$, $f(Y) = Y$

$E(XY) = E(X) \cdot E(Y)$

$\Downarrow$

$Cov(X,Y) = 0$

ii):
$$E[f(X)\cdot g(Y)] = \sum_{x,y} f(x)\cdot g(y)\cdot p(x,y)$$
$$= \sum_{x,y} f(x)\cdot g(y)\cdot \underbrace{p_X(x)\cdot p_Y(y)}_{p(x,y)}$$
$$= \sum_{x,y} f(x)\cdot p_X(x) \cdot g(y)\cdot p_Y(y)$$
$$= \left(\sum_{x} f(x)\cdot p_X(x)\right)\cdot\left(\sum_{y} g(y)\cdot p_Y(y)\right)$$
$$= E[f(X)] \cdot E[g(Y)]$$

Merk:

Hvis X og Y er uavhengige, så er $\mathrm{Cov}(X,Y)=0$.

Men: $\mathrm{Cov}(X,Y)$ kan være null selv om X og Y ikke er uavhengige.

b) Kontinuerlig tilfelle : $\left.\begin{array}{l} X = X(\omega) \\ Y = Y(\omega) \end{array}\right\}$ kont. variable

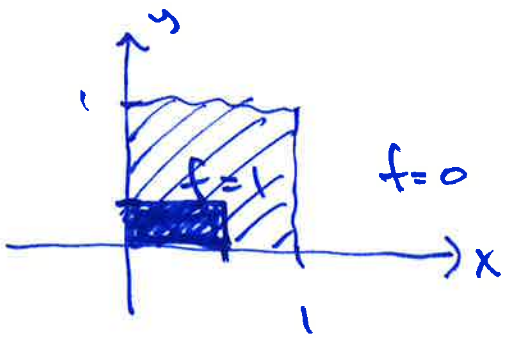
Vi har en simultan sannsynlighetstetthet $f(x,y)$ slik at

$$F(a,b) = P(X \le a, Y \le b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)\, dy\, dx$$

gir den kumulative simultane fordelingsfunksjonen F.

Eks: $f(x,y) = \begin{cases} 1 & , \ 0 \le x \le 1, 0 \le y \le 1 \\ 0 & , \ \text{ellers} \end{cases}$

(uniform fordeling)



$$F(\tfrac{1}{2},\tfrac{1}{3}) = P(X \le 1/2, Y \le 1/3) = \int_0^{1/2}\underbrace{\int_0^{1/3} f(x,y)\, dy}_{1/3}\, dx$$

$$= \int_0^{1/2} 1/3 \, dx = \left[\tfrac{1}{3}x\right]_0^{1/2} = \tfrac{1}{3}\cdot\tfrac{1}{2} - 0 = \underline{\underline{\tfrac{1}{6}}}$$

$$\int_0^{1/3} 1 \, dy = [y]_0^{1/3} = \tfrac{1}{3} - 0 = \underline{1/3}$$

Krav til simultan sannsynlighetstetthet

i) $f(x,y) \geq 0$ for alle x, y

ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$ (dvs $\lim_{\substack{a\to\infty \\ b\to\infty}} F(a,b) = 1$)

Merk:

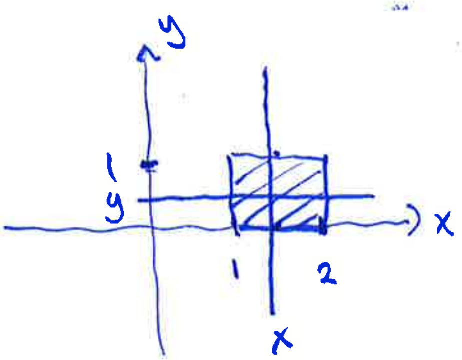
i) $F(a,b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)\,dy\,dx$

ii) $f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = F''_{xy}$

iii) $p(a \leq X \leq a', b \leq Y \leq b') = \int_{a}^{a'}\int_{b}^{b'} f(x,y)\,dy\,dx$

Eks:

$$f(x,y) = \begin{cases} \cancel{4/3} \cdot xy, & 1 \leq x \leq 2, \ 0 \leq y \leq 1 \\ 0, & \text{ellers} \end{cases}$$



i) $f(x,y) \geq 0$ for alle x, y så lenge $k \geq 0$

ii) $\int_{1}^{2}\int_{0}^{1} k \cdot xy\,dy\,dx = 1$

$$\int_{1}^{2}\int_{0}^{1} kxy\,dy\,dx = \int_{1}^{2}\left[kx \cdot \tfrac{1}{2}y^2\right]_0^1 dx$$

$$= \int_{1}^{2} kx \cdot \tfrac{1}{2}\,dx = \left[\tfrac{1}{2}k \cdot \tfrac{1}{2}x^2\right]_1^2$$

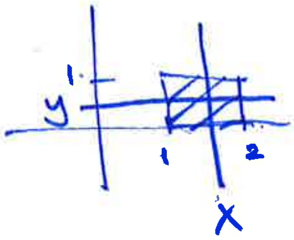
$$= \tfrac{1}{4}k \cdot (4-1) = \tfrac{3}{4}k = 1 \Leftrightarrow k = \tfrac{4}{3}$$

Marginale tettheter:

$f_X(x) = \int_0^1 f(x,y)\,dy$

$f_Y(y) = \int_1^2 f(x,y)\,dx$

$$f(x,y) = \frac{4}{3}xy, \quad 0 \le x \le 2, \ 0 \le y \le 1$$



$$f_X(x) = \int_0^1 \frac{4}{3}xy \, dy = \frac{4}{3}x \cdot \left[\frac{1}{2}y^2\right]_0^1$$

$$= \frac{4}{3}x\left(\frac{1}{2}\right) = \underline{\frac{2}{3}x}, \quad 1 \le x \le 2$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_1^2 x \cdot \frac{2}{3}x \, dx$$

$$= \frac{2}{3} \cdot \left[\frac{1}{3}x^3\right]_1^2 = \frac{2}{9} \cdot (8-1) = \underline{\underline{\frac{14}{9}}}$$

$$f_Y(y) = \int_1^2 \frac{4}{3}xy \, dx = \frac{4}{3}y \cdot \left[\frac{1}{2}x^2\right]_1^2 = \frac{2}{3}y \cdot (4-1)$$

$$= \underline{2y}, \quad 0 \le y \le 1$$

$$E(Y) = \int_0^1 y \cdot 2y \, dy = 2 \cdot \left[\frac{1}{3}y^3\right]_0^1 = \frac{2}{3} \cdot 1 = \underline{\underline{\frac{2}{3}}}$$

$$E(XY) = \int_1^2 \int_0^1 xy \cdot \frac{4}{3}xy \, dy \, dx \quad \left(= \int_0^1 \int_1^2 xy \cdot \frac{4}{3}xy \, dx \, dy \right)$$

$$= \int_1^2 \frac{4}{3}x^2 \left[\frac{1}{3}y^3\right]_0^1 dx = \int_1^2 \frac{4}{3}x^2 \cdot \frac{1}{3} \cdot (1)\, dx$$

$$= \frac{4}{9}\left[\frac{1}{3}x^3\right]_1^2 = \frac{4}{27} \cdot (8-1) = \underline{\underline{\frac{28}{27}}}$$

$$\mathrm{Cor}(X,Y) = E(XY) - E(X) \cdot E(Y) = \frac{28}{27} - \frac{14}{9} \cdot \frac{2}{3} = \underline{\underline{0}}$$

Er X og Y uavhengige?

$$f(x,y) = f_X(x) \cdot f_Y(y)$$
$$\Updownarrow$$
X og Y uavhengige

$$f(x,y) = \begin{cases} \frac{4}{3}xy \,, & 1 \le x \le 2,\ 0 \le y \le 1 \\ 0 \,, & \text{ellers} \end{cases}$$

$$f_X(x) = \begin{cases} \frac{2}{3}x \,, & 1 \le x \le 2 \\ 0 \,, & \text{ellers} \end{cases} \qquad f_Y(y) = \begin{cases} 2y \,, & 0 \le y \le 1 \\ 0 \,, & \text{ellers} \end{cases}$$

Derfor er $f(x,y) = f_X(x) \cdot f_Y(y)$

X og Y er uavh.

Generelt:

$$E[h(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) \cdot f(x,y)\, dy\, dx$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X) \cdot E(Y) \qquad \mathrm{Cov}(X,X) = \mathrm{Var}(X) \ \ldots$$

(de andre regnereglene er også som for det diskrete tilfellet)

Også her gjelder:

X og Y er uavh $\Rightarrow$ i) $\mathrm{Cov}(X,Y) = 0$

ii) $E[f(X) \cdot g(Y)] = E[f(X)] \cdot E[g(Y)]$