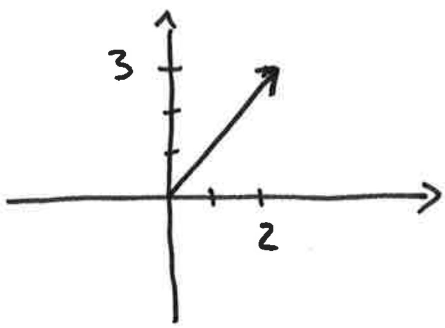
Vektorregning

Tema: - Hva er en vektor?
- Vektorregning og indre produkt (prikkprodukt)
- Lineære underrom

Hva er en vektor?

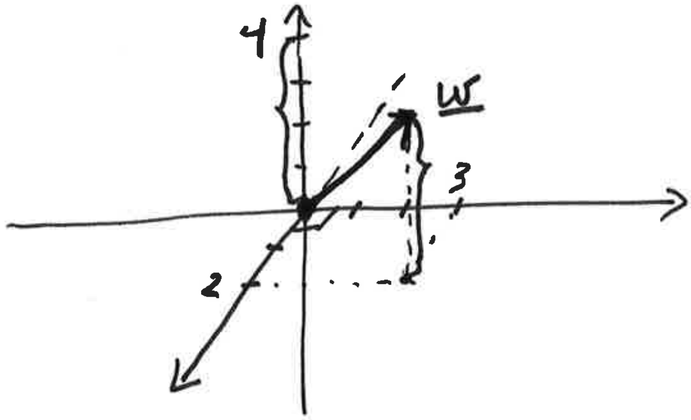
2-vektor

$\underline{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$



En vektor er en forflytning.

3-vektor



$\underline{w} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

En vektor (n-vektor) er et ordnet n-tuppel av tall

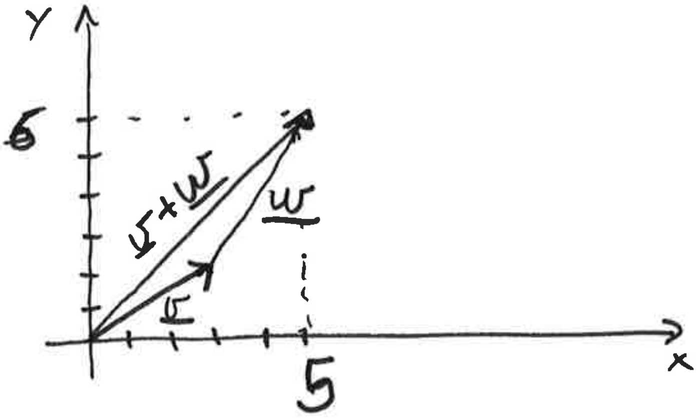
$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ kolonnevektor.

Nullvektoren $\underline{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Addisjon og subtraksjon utføres komponentvis

eks $\underline{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$$\underline{v} + \underline{w} = \begin{pmatrix} 3+2 \\ 2+4 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$



eks.

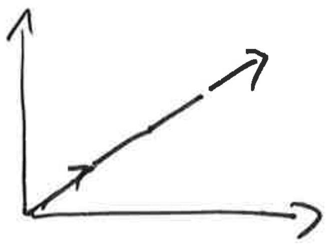
$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$$\underline{v} - \underline{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

Skalarmultiplikasjon (skalar er et tall)

Eks.

$$\underset{\substack{\uparrow \\ \text{skalar}}}{4} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$$



Lineære kombinasjoner

Eks. $\underline{v} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$$2\underline{v} + 3\underline{w} = 2\begin{pmatrix} 5 \\ 1 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \\ 2 \end{pmatrix} + \begin{pmatrix} -6 \\ 9 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 4 \\ 11 \end{pmatrix}}}$$

Eks. $\underline{v} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} -4 \\ 3 \\ 2 \end{pmatrix}$

$$4\underline{v} - 2\underline{w} = \cdots = \begin{pmatrix} 16 \\ -10 \\ -8 \end{pmatrix}$$

For å regne ut lineære kombinasjoner må vektoren ha samme størrelse (like mange elementer).

Lineær kombinasjon av $\underline{v_1}, \underline{v_2}, \dots, \underline{v_m}$ er på formen $r_1 \cdot \underline{v_1} + r_2 \cdot \underline{v_2} + \dots + r_m \cdot \underline{v_m}$

Hver vektor har samme størrelse.

$r_1, \dots, r_m$ er skalarer.

Eks. Hvis $\underline{v_1} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ og $\underline{v_2} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Alle lineær kamb. av $\underline{v_1}$: $r_1 \underline{v_1} = r_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3r_1 \\ 2r_1 \end{pmatrix}$

Alle lineære kamb. av $\underline{v_1}$ og $\underline{v_2}$:

$$r_1 \underline{v_1} + r_2 \underline{v_2} = r_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} + r_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3r_1 - r_2 \\ 2r_1 + 3r_2 \end{pmatrix}$$

Benyttes når vi løser likninger:

** $3\underline{x} + 2\underline{a} = 5\underline{b}$

$3\underline{x} = 5\underline{b} - 2\underline{a}$

* $\underline{x} = \frac{5}{3}\underline{b} - \frac{2}{3}\underline{a}$

Eks. $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\underline{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

* gir $\underline{x} = \frac{5}{3}\begin{pmatrix} 3 \\ 5 \end{pmatrix} - \frac{2}{3} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$= \begin{pmatrix} \frac{15}{3} \\ \frac{25}{3} \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{6}{3} \end{pmatrix} = \begin{pmatrix} \frac{13}{3} \\ \frac{19}{3} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

** kan skrives:

$$3\begin{pmatrix} x \\ y \end{pmatrix} + 2\begin{pmatrix} 1 \\ 3 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

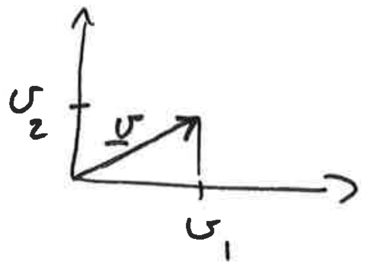
$$\Updownarrow$$

$$\left(\begin{array}{c} 3x + 2 = 15 \\ 3y + 6 = 25 \\ \Updownarrow \\ x = \frac{13}{3} \\ y = \frac{19}{3} \end{array} \right)$$

på "vanlig" måte.

Lengden til en vektor

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$



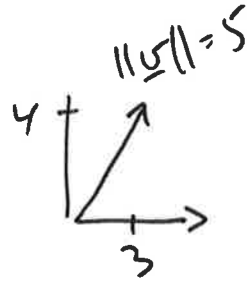
Lengden til $\underline{v}$:

$$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2} \quad \text{(Pytagoras)}$$

<u>Eks</u>

$$\underline{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\|\underline{v}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$



Lengden til en n-vektor

$$\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$ er definert til å være

$$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

$\|\underline{v}\| = 0$ hvis og bare hvis $\underline{v} = \underline{0}$

hviss

$\Longleftrightarrow$

$$\|\underline{v}\| \geq 0$$

Øvelse:

Regn ut lengden av $\underline{v} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 4 \end{pmatrix}$

$$\|\underline{v}\| = \sqrt{2^2 + 3^2 + 1^2 + 4^2} = \underline{\underline{\sqrt{30}}}$$

Indre produkt (skalarprodukt) (prikk produkt)

Eks

$$\underline{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \underline{w} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

$$\underline{v} \cdot \underline{w} = 2 \cdot 5 + 3 \cdot (-1) = 10 - 3 = \underline{\underline{7}}$$

Hvis
$$\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad \underline{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

$$\underline{v} \cdot \underline{w} = v_1 \cdot w_1 + \cdots + v_n \cdot w_n$$

Øvelse:

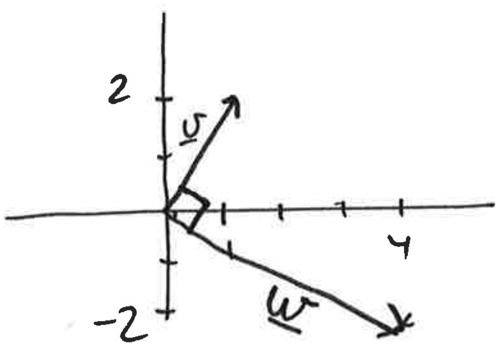
Eks. 1:
$$\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \underline{w} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\underline{v} \cdot \underline{w} = 1 \cdot 2 + 2 \cdot (-1) + 3 \cdot 3 = \underline{\underline{9}}$$

Eks 2:
$$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \underline{w} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\underline{v} \cdot \underline{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \end{pmatrix} = 1 \cdot 4 + 2 \cdot (-2) = 0$$

Tegn $\underline{v}$ og $\underline{w}$ i samme koordinatsystem.



$$\underline{v} \perp \underline{w}$$

Egenskaper ved indreprodukt:

i) $\underline{v} \cdot \underline{w} = \underline{w} \cdot \underline{v}$

ii) $(\underline{v}_1 + \underline{v}_2) \cdot \underline{w} = \underline{v}_1 \cdot \underline{w} + \underline{v}_2 \cdot \underline{w}$

iii) $(r \cdot \underline{v}) \cdot \underline{w} = r\,(\underline{v} \cdot \underline{w})$

iv) $\underline{v} \cdot \underline{v} = \|\underline{v}\|^2$ $\quad \Big(\underline{v} \cdot \underline{v} = v_1^2 + \dots + v_n^2 = \sqrt{v_1^2 + \dots v_n^2}^{\,2} = \|\underline{v}\|^2 \Big)$

v) $\underline{v} \cdot \underline{w} = 0 \Leftrightarrow \underbrace{\underline{v} \perp \underline{w}}_{\text{ortogonale}}$

Cauchy-Schwartz ulikhet

For alle n-vektorer, $\underline{v}$, $\underline{w}$ er:

$$|\underline{v} \cdot \underline{w}| \leq \|\underline{v}\| \cdot \|\underline{w}\|$$

$$\Updownarrow$$

$$\frac{|\underline{v} \cdot \underline{w}|}{\|\underline{v}\| \cdot \|\underline{w}\|} \leq 1$$

$$\Updownarrow$$

$$-1 \leq \frac{\underline{v} \cdot \underline{w}}{\|\underline{v}\| \cdot \|\underline{w}\|} \leq 1$$

Eks $\quad \underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \underline{w} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$\left.\begin{aligned} \underline{v} \cdot \underline{w} &= 1 \cdot 3 + 2 \cdot 1 = 5 \\ \|\underline{v}\| &= \sqrt{1^2 + 2^2} = \sqrt{5} \\ \|\underline{w}\| &= \sqrt{3^2 + 1^2} = \sqrt{10} \end{aligned}\right\} \Rightarrow \frac{\underline{v} \cdot \underline{w}}{\|\underline{v}\| \cdot \|\underline{w}\|} = \frac{5}{\sqrt{5}\sqrt{10}} = \frac{1}{\sqrt{2}} \approx 0{,}7$$

Bevis ~~[illegible]~~ 7 x.

C-S: $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$$|\underline{u} \cdot \underline{v}| = |u_1 v_1 + u_2 v_2| = \sqrt{(u_1 v_1 + u_2 v_2)^2}$$

$$= \sqrt{u_1^2 v_1^2 + \underbrace{2 u_1 v_1 u_2 v_2}_{} + v_2^2 u_2^2}$$

$$\|v_1\| \cdot \|v_2\| = \sqrt{v_1^2 + v_2^2} \cdot \sqrt{u_1^2 + u_2^2}$$

$$= \sqrt{(v_1^2 + v_2^2)(u_1^2 + u_2^2)}$$

$$= \sqrt{v_1^2 u_1^2 + \underbrace{v_1^2 u_2^2 + v_2^2 u_1^2}_{} + v_2^2 u_2^2}$$

Nok å vise at $2 u_1 v_1 u_2 v_2 < v_1^2 u_2^2 + v_2^2 u_1^2$

$$\Updownarrow$$

$$(v_1 u_2)^2 + 2 (v_1 u_2)(u_1 v_2) + (u_1 v_2)^2 \geq 0$$

$$\Updownarrow$$

$$(v_1 u_2 + u_1 v_2)^2 \geq 0$$

~~[illegible]~~

som alltid stemmer!

Korrelasjon

(fra statistikk : $r = \dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2}\sqrt{\sum (y_i - \bar{y})^2}}$)

To kolonner:

$$\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{array} \qquad \underline{x} = \begin{pmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{pmatrix} \qquad \underline{y} = \begin{pmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix}$$

$$r = \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\| \cdot \|\underline{y}\|}$$

Dermed er $-1 \le r \le 1$

Eks.

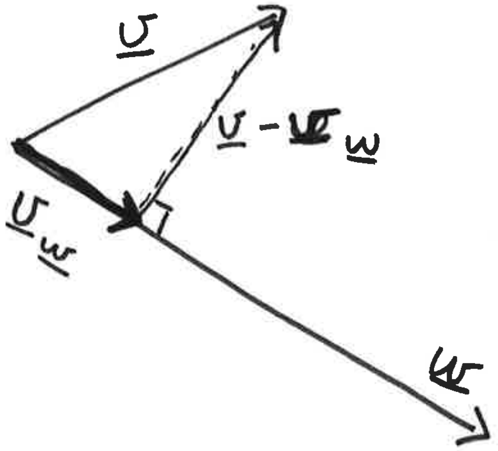
x_i	y_i
2	8
3	11
5	12
7	15
3	9
$\bar{x} = 4$	$\bar{y} = 11$

$$\underline{x} = \begin{pmatrix} -2 \\ -1 \\ 1 \\ 3 \\ -1 \end{pmatrix} \qquad \underline{y} = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 4 \\ -2 \end{pmatrix}$$

$$\left.\begin{array}{l} \underline{x} \cdot \underline{y} = 6 + 0 + 1 + 12 + 2 = 21 \\ \|\underline{x}\| = \sqrt{4+1+1+9+1} = \sqrt{16} = 4 \\ \|\underline{y}\| = \sqrt{9+0+1+16+4} = \sqrt{30} \end{array}\right\} \Rightarrow r = \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\| \cdot \|\underline{y}\|}$$

$$= \frac{21}{4 \cdot \sqrt{30}} = \underline{\underline{0{,}9585}}$$

Projeksjon



$\text{proj}_{\underline{w}}(\underline{v}) = \underline{v}_{\underline{w}}$ er den vektoren som er parallell med $\underline{w}$ og slik at

$$\boxed{\begin{array}{c} \underline{v}_{\underline{w}} \perp \underline{v} - \underline{v}_{\underline{w}} \\ \Updownarrow \\ \underline{v}_{\underline{w}} \cdot (\underline{v} - \underline{v}_{\underline{w}}) = 0 \end{array}}$$

og

$$\boxed{\begin{array}{c} \underline{v}_{\underline{w}} \parallel \underline{w} \\ \Updownarrow \\ \underline{v}_{\underline{w}} = k \cdot \underline{w} \end{array}} \quad *$$

($\parallel$ "parallell med")

$$\underline{v}_{\underline{w}} \cdot (\underline{v} - \underline{v}_{\underline{w}}) = 0$$

$$k \cdot \underline{w} \cdot (\underline{v} - k\underline{w}) = 0 \qquad | : k$$

$$\underline{w} \cdot \underline{v} - k \cdot \underline{w} \cdot \underline{w} = 0$$

$$k = \frac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}} = \frac{\underline{v} \cdot \underline{w}}{\|\underline{w}\|^2}$$

$*$ gir $\text{proj}_{\underline{w}}(\underline{v}) = \boxed{\underline{v}_{\underline{w}} = \frac{\underline{v} \cdot \underline{w}}{\|\underline{w}\|^2} \cdot \underline{w}}$

__Eks.__ $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Finn $\underline{v}_{\underline{w}}$

$$\underline{v}_{\underline{w}} = \frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix}}{\left\|\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\|^2} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{5}{\sqrt{10}^2} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}}}$$

Lineære underrom neste forelesning