

# FORELESNING 12

ELE3719 BI

EIVIND ERIKSEN, APR 12 2016

MATEMATIKK

Plan:

- ① Separable diff. likninger (fortsettelse)
- ② Lineære første ordus diff. likninger
- ③ Stabilitet (neste gang)
- ④ Systemer av diff. likninger

Prøver:

[D.H.] Del 1-3

## ① Separable diff. likninger

Separabel: kan skrives

$$y' = f(y) \cdot g(t)$$

$$\frac{1}{f(y)} y' = g(t)$$

$$\int \frac{1}{f(y)} (y' dt) = \int g(t) dt$$

$$\int \frac{1}{f(y)} dy = \int g(t) dt$$

Eks:

$$y' = \frac{t}{y}$$

$$y' = \frac{1}{y} \cdot t$$

↑      ↑  
f(y)   g(t)

ok separabel

$$\Rightarrow y y' = t$$

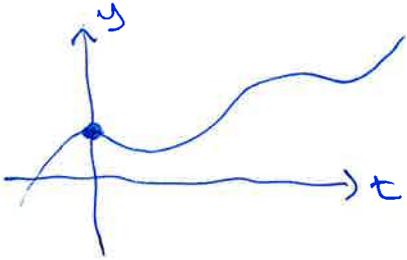
$$\int y y' dt = \int t dt$$

$$\int y dy = \int t dt$$

$$\frac{1}{2} y^2 + C_1 = \frac{1}{2} t^2 + C_2$$

$$y' = \frac{t}{y}$$

$$y(0) = 1 \Leftrightarrow t=0, y=1$$



$$y = \pm \sqrt{t^2 + 2K}$$

$$(1)^2 = \left( \pm \sqrt{0^2 + 2K} \right)^2$$

$$1 = 2K \Rightarrow K = 1/2$$

$\Downarrow$

$$y = \underline{\underline{\sqrt{t^2 + 1}}}$$

$$\frac{1}{2}y^2 = \frac{1}{2}t^2 + \underbrace{C_2 - C_1}$$

$$\frac{1}{2}y^2 = \frac{1}{2}t^2 + K \quad | \cdot 2$$

$$y^2 = t^2 + 2K$$

$$y = \pm \sqrt{t^2 + 2K}$$



generell Lösung

$$\pm \sqrt{t^2 + 2(C_2 - C_1)}$$

$$y = \pm \sqrt{t^2 + K'}$$

$$\underline{t=0, y=1}: \quad 1 = \pm \sqrt{0^2 + K'}$$

$$1 = K'$$

$$y = \underline{\underline{\sqrt{t^2 + 1}}}$$

Eks:  $y' = y(t+y+t+1)$

$$y' = (y+1) \cdot (t+1)$$

$$\frac{1}{y+1} y' = t+1 \Rightarrow \int \frac{1}{y+1} dy = \int (t+1) dt$$

$$\ln |y+1| = \frac{1}{2}t^2 + t + C$$

$$|y+1| = e^{\frac{1}{2}t^2 + t + C}$$

$$y+1 = \pm e^{\frac{1}{2}t^2 + t + C}$$

$$y = \pm e^{\frac{1}{2}t^2 + t + C} - 1$$

$$= \pm e^{\frac{1}{2}t^2 + t} \cdot e^C - 1$$

$$y = \underline{\underline{K e^{\frac{1}{2}t^2 + t} - 1}} \quad (K = \pm e^C)$$

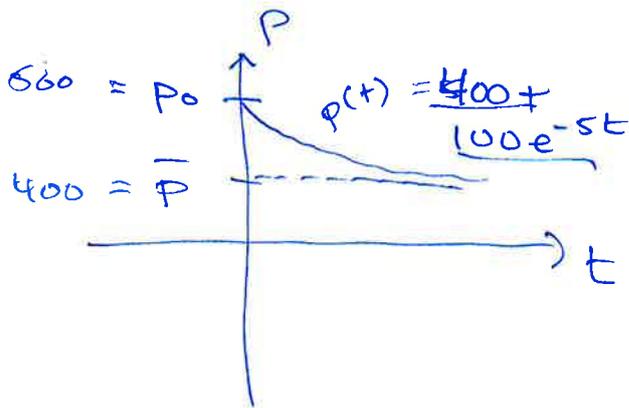
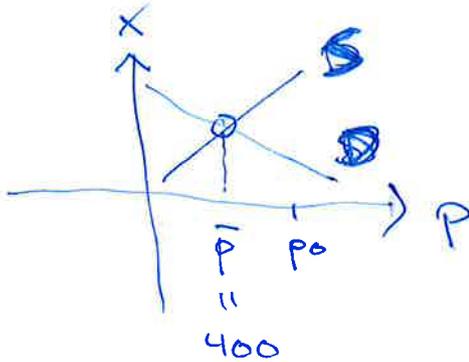
implizit  
Lösung.

Eqs:

$$D = \alpha - \beta p = 5000 - 4p$$

$$S = a + bp = 1000 + 6p$$

$$p' = \lambda \cdot (D - S) = 0.5(D - S), \quad p(0) = 500$$



$$p' = 0.5 \cdot ((5000 - 4p) - (1000 + 6p))$$

$$= 0.5(4000 - 10p)$$

$$p' = 2000 - 5p \quad p = p(t)$$

$$p' = 5 \cdot (400 - p)$$

$$\frac{1}{400 - p} p' = 5$$

$$\int \frac{1}{400 - p} \overset{dp}{p'} dt = \int 5 dt$$

$$\frac{1}{-1} \ln |400 - p| = 5t + C$$

$$\ln |400 - p| = -5t - C$$

$$|400 - p| = e^{-5t - C}$$

$$400 - p = \pm e^{-5t - C}$$

$$= \pm e^{-5t} \cdot e^{-C}$$

$$400 - p = K e^{-5t} \quad (K = \pm e^{-C})$$

$$p = 400 - K e^{-5t}$$

$$\leftarrow 400 - K e^{-5t} = p$$

$$p(0) = 500: \quad t=0, \quad p=500$$

$$500 = 400 - K \cdot e^0$$

$$K = -100$$

$$p(t) = 400 + 100 e^{-5t}$$

② Lineære første ordns diff. ligninger

Defn: En første ordens diff. ligning er lineær hvis den kan skrives

$$y' = b(t) - a(t) \cdot y \leftarrow (\text{lineær i } y)$$

$$\Updownarrow$$

$$y' + a(t) \cdot y = b(t)$$

der  $a(t), b(t)$  er udtrykk i  $t$ .

Ex:  $y' + 2y = 3$       lineær  $\begin{cases} a(t) = 2 \\ b(t) = 3 \end{cases}$   
( $a(t), b(t)$  konstanter)

$y' + ty = e^t$       lineær  $\begin{cases} a(t) = t \\ b(t) = e^t \end{cases}$

$y' + ty^2 = t^2$       ikke lineær

Ex:  $y' + y = 4$       lineær ( $a=1, b=4$ )

$y' = 4 - y = (4 - y) \cdot 1$       separabel ( $f(y) = 4 - y, g(t) = 1$ )

# Lösungsmethoden für lineare diff. Gleichungen

$$y' + a(t) \cdot y = b(t)$$

## ① Integrierende Faktor

Ex:  $y' + 2y = 3 \quad | \cdot u$   
 $y' \cdot u + 2y \cdot u = 3u$   
 $y' \cdot u + y \cdot u' \rightarrow (y \cdot u)' = 3u$

$u$ : integrierende Faktor

Gesucht:

$$2u = u'$$

$$u = e^{2t}$$

$$u' = e^{2t} \cdot 2 = 2u$$

$$y' + 2y = 3 \quad | \cdot e^{2t}$$

$$y' \cdot e^{2t} + 2y \cdot e^{2t} = 3e^{2t}$$

$$y' \cdot e^{2t} + y \cdot e^{2t} \cdot 2 \rightarrow (y \cdot e^{2t})' = 3e^{2t} \quad | \int \dots dt$$

$$y \cdot e^{2t} = \int 3e^{2t} dt$$

$$y \cdot e^{2t} = 3 \cdot \frac{1}{2} e^{2t} + C$$

$$y = \frac{\frac{3}{2} e^{2t} + C}{e^{2t}} = \underline{\underline{\frac{3}{2} + C \cdot e^{-2t}}}$$

Generelt:

a) Konstante koeffisienter:

$$y' + ay = b$$

Integrerende faktor:

$$u = e^{at}$$

(fordi  $u' = a \cdot u$ )

Steg:

$$y' + ay = b \quad | \cdot u = e^{at}$$

$$y'e^{at} + aye^{at} = be^{at}$$

$$(y \cdot e^{at})' = be^{at}$$

$$y \cdot e^{at} = \int be^{at} dt$$

$$y \cdot e^{at} = b \cdot \frac{1}{a} e^{at} + C$$

$$y = \frac{\frac{b}{a} e^{at} + C}{e^{at}}$$

Formel:

(gjelder når  $a(t) = a$ ,  $b(t) = b$  er konstanter)

$$y = \frac{b}{a} + C \cdot e^{-at}$$

Ex:

$$y' + y = 4$$

linear  
konst. koeff.

$$a = 1 \\ b = 4$$

$\Leftrightarrow$

$$y = \underline{4 + C \cdot e^{-t}} = \frac{b}{a} + C \cdot e^{-at}$$

$\uparrow$

$$\frac{b}{a} = \frac{4}{1}$$

$$-at = -1 \cdot t = -t$$

b) Linje konstante koefficienter:  $y' + a(t) \cdot y = b(t)$

Exs:  $y' + 2t \cdot y = t$   $\begin{cases} a(t) = 2t \\ b(t) = t \end{cases}$

Integrerande faktor:

Må finne  $u$  s.a.  $u' = a(t) \cdot u$

Vilken konstant:

$$e^{\int a(t) dt}$$

$a(t) = 2t$ :

$$\int a(t) dt = \int 2t dt = t^2 + C$$

$$u = e^{\int a(t) dt} = e^{t^2 + C} = e^{t^2} \quad (C=0)$$

$a(t) = 2$ :

$$\int 2 dt = 2t$$

$$u = e^{2t}$$

Konklusion:

Integrerande faktor er  
 $u = e^{\int a(t) dt}$

Exs:

$$y' + \underbrace{(2t)}_{a(t)} y = \underbrace{(t)}_{b(t)} \quad | \cdot e^{t^2}$$

Integrerande faktor:

$$a(t) = 2t$$

$$\int a(t) dt = \int 2t dt = t^2 + C$$

$$u = e^{t^2} \quad (C=0)$$

$$y' \cdot e^{t^2} + 2t \cdot y \cdot e^{t^2} = t \cdot e^{t^2}$$

$$(y \cdot e^{t^2})' = t \cdot e^{t^2}$$

$$y \cdot e^{t^2} = \int t \cdot e^{t^2} dt$$

$$v = t^2 \\ dv = 2t \cdot dt$$

$$= \int t \cdot e^v \cdot \frac{dv}{2t} = \frac{1}{2} \int e^v dv$$

$$= \frac{1}{2} \cdot e^v + C = \frac{1}{2} e^{t^2} + C$$

$$y = \frac{1}{2} + C \cdot e^{-t^2}$$

$$\frac{y \cdot e^{t^2}}{e^{t^2}}$$

Formel:  $y' + a(t)y = b(t)$

$$y = \frac{1}{e^{\int a(t) dt}} \cdot \int e^{\int a(t) dt} \cdot b(t) dt$$

$u = e^{\int a(t) dt}$ :  $y = \frac{1}{u} \cdot \int u \cdot b(t) dt$

② "Superposition principle"

$$y' + a(t) \cdot y = b(t)$$

Linear diff. lösning av första ord.

Superposition principle:  $y = y_h + y_p$

a)  $y_h$ : Den generella lösningen av  $y' + a(t)y = 0$   
Diff. lön.  $y' + a(t)y = 0$  kallas homogen.

Formel:  
 $y = \frac{b}{a} + c \cdot e^{-at}$   
 $= c \cdot e^{-at}$

$a(t) = a$  konstant:

$y = e^{rt}$

$y' + ay = 0$

$r \cdot e^{rt} + a \cdot e^{rt} = 0$

$(r+a) e^{rt} = 0$

$r+a = 0$

$\Rightarrow r = -a$

Karakteristisk lösning

$y_h = C e^{-at}$

Metoden med karakteristisk lösning:

$y' + ay = 0$

Kar. lösning:

$\rightarrow r+a = 0 \Rightarrow r = -a \Rightarrow y_h = C e^{-at}$

b)  $y_p$ : En partikulær (speciell) løsning  
av  $y' + a(t)y = b(t)$ .

Ekko:  $y' - 3y = 2e^t$  linear  $a(t) = -3$   
 $b(t) = 2e^t$

$$y = y_h + y_p = \underline{\underline{C \cdot e^{3t} + e^t}}$$

$y_h$ :  $y' - 3y = 0$   
 $r - 3 = 0 \Rightarrow r = 3 \Rightarrow y_h = \underline{C \cdot e^{3t}}$

$y_p$ :  $y' - 3y = 2e^t$  Greiter  $\left\{ \begin{array}{l} y = A \cdot e^t \\ y' = A \cdot e^t \end{array} \right.$

$$(Ae^t) - 3(Ae^t) = 2e^t$$

$$\frac{Ae^t - 3Ae^t}{e^t} = \frac{2e^t}{e^t}$$

$$A - 3A = 2$$

$$-2A = 2 \quad \underline{A = -1} \Rightarrow y_p = \underline{-e^t}$$

# Hvordan "gætter" man $y_p$ ?

Ek:  $y' + ay = \textcircled{b}$

Gætter:  $y = A$   $\leftarrow$  af samme type  
- med nogen parameter som kan justeres

$$y' = 0$$

$\Leftrightarrow$

$$0 + a \cdot A = b$$

$$A = \frac{b}{a} \Rightarrow y_p = \underline{\underline{\frac{b}{a}}}$$

Formel:  $y = \frac{b}{a} + \underbrace{C \cdot e^{-at}}_{y_h}$

Ek:  $y' + 3y = \textcircled{3e^{-t}}$

$$y = y_h + y_p = \underline{\underline{C \cdot e^{-3t} + \frac{3}{2}e^{-t}}}$$

$y_h$ :  $y' + 3y = 0$   
 $r + 3 = 0$   
 $r = -3 \rightarrow y_h = \underline{C \cdot e^{-3t}}$

$y_p$ :  $y = \underline{A \cdot e^{-t}}$   
 $y' = A \cdot e^{-t} \cdot (-1)$

$$(-Ae^{-t}) + 3 \cdot (Ae^{-t}) = 3e^{-t}$$
$$-A + 3A = 3$$
$$2A = 3 \rightarrow A = \underline{\underline{\frac{3}{2}}}$$

$b(t) = 3e^{-t}$   
 $b'(t) = 3 \cdot e^{-t} \cdot (-1)$   
Samme type:  $A \cdot e^{-t}$

$$\rightarrow y_p = \frac{3}{2} \cdot e^{-t}$$

Generelt:  $y' + ay = b$

$$y = y_h + y_p = C e^{-at} + \frac{b}{a}$$

$y_h$ :  $y' + ay = 0$

$$r + a = 0$$

$$r = -a$$

$$y_h = C \cdot e^{-at}$$

$y_p$ :  $y' + ay = b$

$$y_p = \frac{b}{a}$$

Hverken fungerer "superposition principle":  $y = y_h + y_p$ ?

Lineær diff. ligning:  $y' + a(t)y = b(t)$

Definer funktional:  $J(y) = y' + a(t) \cdot y$

Ekse:  $J(y) = y' + 2t \cdot y$  med  $a(t) = 2t$

$$J(t) = 1 + 2t \cdot t = 1 + 2t^2$$

$$J(1+t) = 1 + 2t \cdot (1+t) = 1 + 2t + 2t^2$$

Egenskaber ved  $J(y) = y' + a(t) \cdot y$

i)  $J(y_1 + y_2) = J(y_1) + J(y_2)$  ←

ii)  $J(c \cdot y_1) = c \cdot J(y_1)$  (for  $c$  konst)

$$\begin{aligned} J(y_1 + y_2) &= (y_1 + y_2)' + a(t) \cdot (y_1 + y_2) \\ &= y_1' + y_2' + a(t)y_1 + a(t)y_2 \\ &= J(y_1) + J(y_2) \end{aligned}$$

Vet at  $J(y_h) = 0$ ,  $J(y_p) = b(t)$ .

Da følger det at:

$$J(y_h + y_p) = J(y_h) + J(y_p) = 0 + b(t) = b(t) \Rightarrow y_h + y_p \text{ er løsn.}$$

$$y \text{ løsn} \Rightarrow J(y) = b(t) \Rightarrow J(y - y_p) = b(t) - b(t) = 0$$

$$\Rightarrow y - y_p = y_h \Rightarrow y = y_h + y_p.$$

### ③ Systemer av lineære diff. likninger

Ekso: 
$$\left. \begin{aligned} x' &= x + 2y \\ y' &= 4x - y \end{aligned} \right\} \text{ Skal finne } x(t), y(t)$$

Exo: 
$$\begin{aligned} x' &= x \\ y' &= -y \end{aligned}$$

(1)  $x' = x$

$x' - x = 0$

linear  $a = -1, b = 0$

$$x(t) = \frac{b}{a} + C \cdot e^{-at}$$

$$= 0 + C \cdot e^t = \underline{C_1 e^t}$$

(2)  $y' = -y$

$y' + y = 0$

linear  $a = 1, b = 0$

$$y(t) = \frac{b}{a} + C \cdot e^{-at} = \underline{C_2 e^{-t}}$$

Løsning: 
$$\begin{aligned} x(t) &= C_1 e^t \\ y(t) &= C_2 e^{-t} \end{aligned}$$

Skrive måte ved matriser:

$$\begin{aligned} x' &= x \\ y' &= -y \end{aligned}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Leftrightarrow$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C_1 e^t \\ C_2 e^{-t} \end{pmatrix} = C_1 \cdot \begin{pmatrix} e^t \\ 0 \end{pmatrix} + C_2 \cdot \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}$$

Ekse:

$$x' = x + 2y$$

$$y' = 4x - y$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

A =  $\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$ ; Egenverdier og egenvektorer

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) \cdot (-1-\lambda) - 8 = 0$$

$$\lambda^2 - 1 - 8 = 0$$

$$\lambda^2 - 9 = 0 \Rightarrow \lambda = \underline{\underline{3, -3}}$$

$\lambda = 3$ :

$$\begin{pmatrix} -2 & 2 \\ 4 & -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -2x + 2y &= 0 \\ \cancel{4x} - \cancel{4y} &= 0 \end{aligned}$$

$$y \text{ fri, } x = y$$

$$\begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{\underline{x}} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \underline{\underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}}$$

$\lambda = -3$ :

$$\begin{pmatrix} 4 & 2 \\ \cancel{4} & \cancel{-2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$4x + 2y = 0$$

$$x \text{ fri}$$

$$y = -2x$$

$$\underline{\underline{z}} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -2x \end{pmatrix} = x \cdot \underline{\underline{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = A \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\underline{x}' = A \cdot \underline{x}$$

$$P \cdot \underline{u}' = A \cdot P \cdot \underline{u}$$

$$\underline{u}' = P^{-1} A P \cdot \underline{u}$$

$$\underline{u}' = D \cdot \underline{u}$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}$$

$$u' = 3u \rightarrow u' - 3u = 0$$

$$u = c_1 e^{3t}$$

$$v' = -3v$$

||

$$u = c_1 \cdot e^{3t}$$

$$v = c_2 \cdot e^{-3t}$$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}, P = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \underline{v}_1 & \underline{v}_2 \end{pmatrix}$$

$$\boxed{P^{-1} A P = D}$$

Variaablsubstitution:  $P^{-1} = \frac{1}{-3} \cdot \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix}$

$$\underline{x} = P \cdot \underline{u} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\underline{u} = P^{-1} \cdot \underline{x}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = P^{-1} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$u = \frac{1}{3} (2x - y) = \frac{2}{3}x - \frac{1}{3}y$$

$$v = \frac{1}{3} (x - y) = \frac{1}{3}x - \frac{1}{3}y$$

$$\underline{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = (P \cdot \underline{u})' = P \cdot \underline{u}'$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} c_1 e^{3t} \\ c_2 e^{-3t} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{3t} + c_2 e^{-3t} \\ c_1 e^{3t} - 2c_2 e^{-3t} \end{pmatrix}$$

$$= c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdot e^{-3t}$$

$$\uparrow$$

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_1 = 3$$

$$\uparrow$$

$$\underline{v}_2$$

$$\lambda_2 = -3$$

General solution:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

||

$$\underline{\underline{\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \underline{v}_1 \cdot e^{\lambda_1 t} + C_2 \underline{v}_2 \cdot e^{\lambda_2 t}}}$$

where  $C_1, C_2$  are constants,

$\lambda_1, \lambda_2$  are eigenvalues of  $A$

$\underline{v}_1, \underline{v}_2$  are eigenvectors (for  $\lambda_1, \lambda_2$ )