Plan

1 Simultant fordelte kontinuerlige stokastiske variabler
2 Kumulativ fordelingsfunksjon

1 Simultant fordelte stokastiske variabler: X, Y

* ett forsøk (stokastisk)
* både X og Y avhenger av utfallet

Eks: Vi kaster to terninger

X = summen 2, 3, ..., 36

Y = største − minste 0, 1, ..., 5

Eks: Vi trekker en tilfeldig børsdag

X = avkastning til hovedindeksen

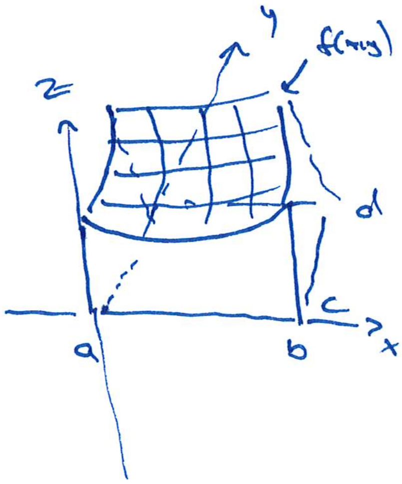
Y = " — SP500

Simultan sannsynlighetstetthet: $f(x,y)$

$$P(a \le X \le b \text{ og } c \le Y \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$\parallel$$

$$P(a \le X \le b, c \le Y \le d)$$

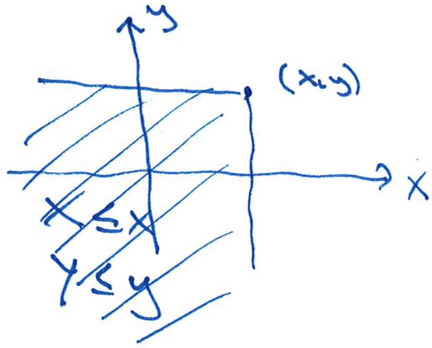


Krav: i) $f(x,y) \ge 0$ for alle x, y

ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$

② Fordelingsfunksjon

$$F(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dy\,dx$$



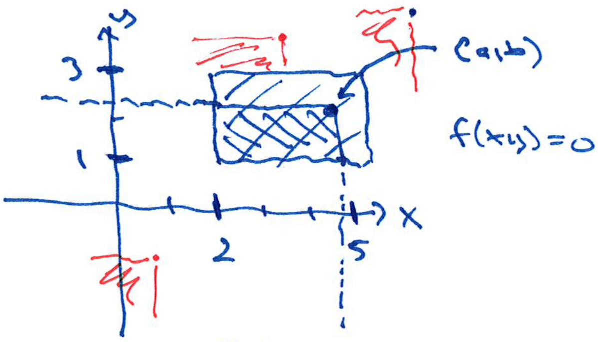
$f(x,y) \longrightarrow F(x,y)$

simultan sanns.tetthet

$$f(x,y) = F''_{xy} \longleftarrow F(x,y)$$

$$F''_{xy} = F''_{yx}$$

Eks: $f(x,y) = \begin{cases} \frac{1}{91}xy^2, & 2 \le x \le 5,\ 1 \le y \le 3 \\ 0, & \text{ellers} \end{cases}$



i) $f(x,y) \ge 0$ ok

ii) $\iint f(x,y)\,dy\,dx = 1$

$$\int_2^5\int_1^3 \frac{1}{91}xy^2\,dy\,dx$$

$$= \frac{1}{91}\int_2^5 x\left[\tfrac{1}{3}y^3\right]_1^3 dx$$

$$= \frac{1}{91}\int_2^5 x(9-1/3)\,dx$$

$$= \frac{1}{91}\cdot\frac{26}{3}\left[\tfrac{1}{2}x^2\right]_2^5$$

$$= \frac{1}{91}\cdot\frac{26}{3}\cdot\frac{1}{2}(25-4)$$

$$= \frac{1}{91}\cdot\frac{26}{6}\cdot 21 = \frac{13\cdot 7}{91} = 1 \quad \text{ok}$$

$$F(a,b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)\,dy\,dx$$

$$= \int_2^a\int_1^b \frac{1}{91}xy^2\,dy = \frac{1}{91}\int_2^a x\left[\tfrac{1}{3}y^3\right]_1^b dx$$

$$= \frac{1}{91}\int_2^a x\left(\tfrac{1}{3}b^3 - \tfrac{1}{3}\right)dx$$

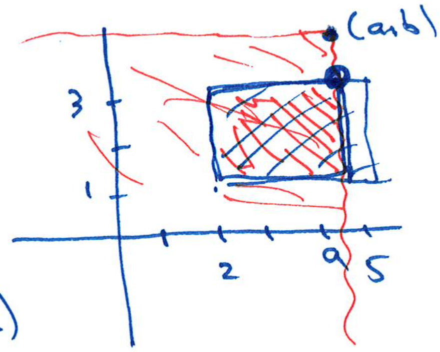
$$= \frac{1}{91\cdot 3}\int_2^a x\cdot(b^3-1)\,dx$$

$$= \frac{b^3-1}{91\cdot 3}\left[\tfrac{1}{2}x^2\right]_2^a = \frac{(b^3-1)(a^2-4)}{91\cdot 6} = \frac{(a^2-4)(b^3-1)}{546}, \quad 2 \le a \le 5,\ 1 \le b \le 3$$

(a,b) slik at $2 \le a \le 5$, $b \ge 3$:

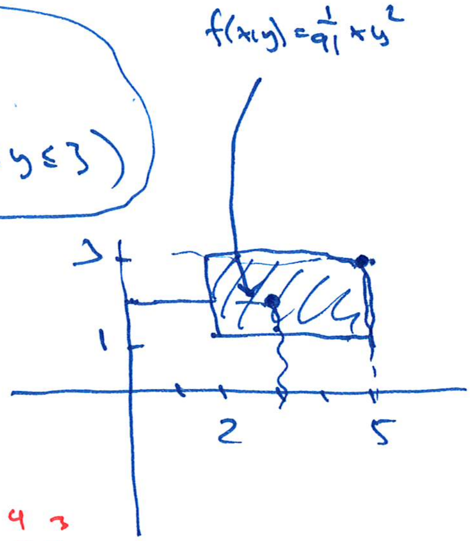
$$F(a,b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)\,dy = \int_{2}^{a}\int_{1}^{3} \frac{1}{91} xy^2\,dy\,dx$$

$$= F(a,3) = \frac{(a^2-4)(3^3-1)}{546} = \underline{\underline{\frac{26}{546}(a^2-4)}}$$



Konklusjon:

$$F(x,y) = \frac{(x^2-4)(y^3-1)}{546}, \quad (2 \le x \le 5,\ 1 \le y \le 3)$$



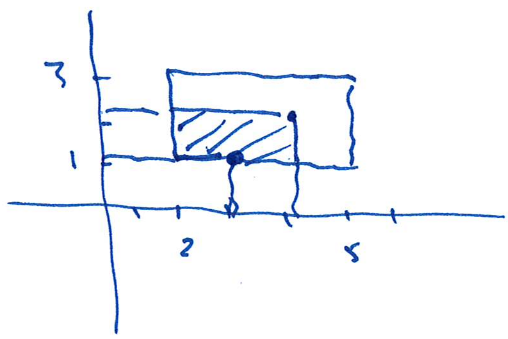
$$F(5,3) = p(X \le 5, Y \le 3) = 1$$

$$\frac{21 \cdot 26}{546} = \frac{3 \cdot 7 \cdot 2 \cdot 13}{6 \cdot 91} = \frac{6 \cdot 7 \cdot 13}{6 \cdot 7 \cdot 13} = 1$$

$$p(X \le 3, Y \le 2) = F(3,2)$$

$$= \frac{5 \cdot 7}{546} = \frac{35}{546} = \underline{\underline{\frac{5}{78}}}$$

Alt: $\displaystyle\int_3^4\int_2^3 \frac{1}{91} xy^2\,dy\,dx$



$$p(3 \le X \le 4, 1 \le Y \le 2)$$

$$= p(3 \le X \le 4, Y \le 2)$$

$$= F(4,2) - 0 = \frac{12 \cdot 7}{546} = \underline{\underline{\frac{12}{78}}}$$

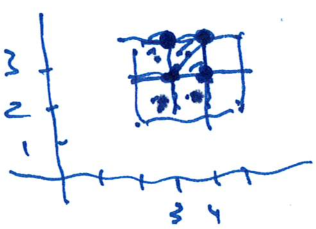
$F(4,2) - F(3,1)$

$$p(3 \le X \le 4, 2 \le Y \le 3)$$

$$= F(4,3) - F(4,2) - F(3,3) + F(3,2)$$

$$= \frac{12 \cdot 26}{546} - \frac{12 \cdot 7}{546} - \frac{5 \cdot 26}{546} + \frac{5 \cdot 7}{546}$$

$$= \underline{\underline{133/546}}$$



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$$F(x,y) = \frac{(x^2-4)(y^3-1)}{546}, \quad 2 \le x \le 5, \; 1 \le y \le 3$$

$$F''_{xy} = \left(F'_x\right)'_y = \left(\frac{y^3-1}{546} \cdot 2x\right)'_y = \frac{2x}{546} \cdot 3y^2$$

$$= \frac{6xy^2}{546} = \frac{\cancel{6}xy^2}{91 \cdot \cancel{6}} = \underline{\underline{\frac{1}{91}xy^2}}$$

$$F(x,y) = F(x,3)$$

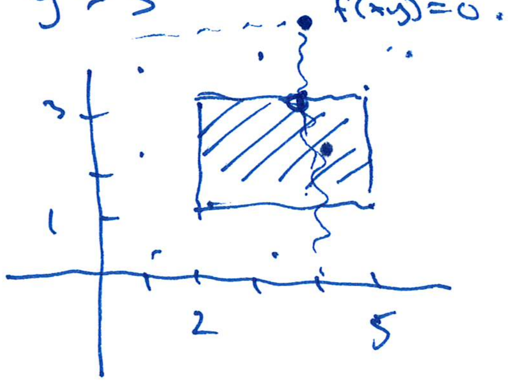
$$= \frac{(x^2-4) \cdot 26}{546}$$

$$= \frac{26}{546}(x^2-4)$$

$$F''_{xy} = 0$$

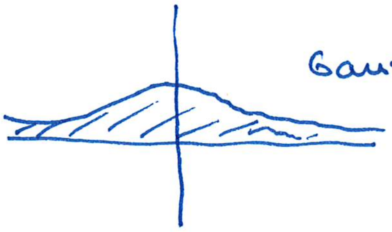
$2 \le x \le 5$,

$y > 3$



Eks: Dobbelt integral

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



Gauss / std. normalfordeling

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Updownarrow$$

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

$$\boxed{\begin{array}{l} x = \sqrt{2}\cdot u \\ dx = \sqrt{2}\,du \end{array}} \qquad \begin{array}{l} u = x/\sqrt{2} \\ du = dx/\sqrt{2} \end{array}$$

$$\int_{-\infty}^{\infty} e^{-u^2}\sqrt{2}\,du = \sqrt{2\pi}$$

$$\Updownarrow$$

$$\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$$

Gauss-integralet:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\cdot\int_0^{\infty} e^{-x^2}\,dx$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \cdot \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$= 2\int_0^{\infty} e^{-x^2}\,dx \cdot 2\int_0^{\infty} e^{-y^2}\,dy$$

$$= 4\cdot\int_0^{\infty}\int_0^{\infty} e^{-x^2}e^{-y^2}\,dy\,dx$$

$$t = y/x \qquad dt = dy/x \qquad \boxed{\begin{array}{l} y = xt \\ dy = x\,dt \end{array}}$$

$$= 4\int_0^{\infty}\int_0^{\infty} e^{-x^2-x^2t^2}\,dy\,dx$$

$$= 4\int_0^\infty \int_0^\infty x \, e^{-x^2(1+t^2)} \, dt \, dx$$

$$= 4\int_0^\infty \int_0^\infty x \, e^{-x^2(1+t^2)} \, dx \, dt$$

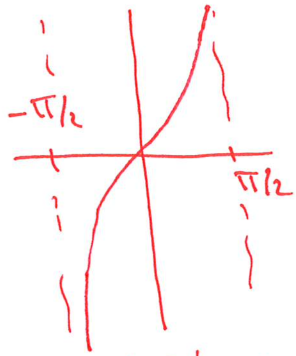
$$= 4\int_0^\infty \left[e^{-x^2(1+t^2)} \cdot \frac{1}{-2(1+t^2)} \right]_0^\infty dt$$

$$= 4\int_0^\infty -\frac{1}{2(1+t^2)} (0-1) \, dt$$

$$= \frac{4}{2}\int_0^\infty \frac{1}{1+t^2} \, dt = 2\int_0^\infty \frac{1}{1+t^2} \, dt$$

$$= 2\left[\arctan(t)\right]_0^\infty = 2\left(\pi/2 - 0\right) = \pi$$

$$I^2 = \pi \implies I = \sqrt{\pi}$$



$y = \tan x = \dfrac{\sin x}{\cos x}$

$\Updownarrow$

Omvendt fn.

$y = \arctan x$
$= \tan^{-1} x$