Plan

1 Marginalene
2 Forventning, varians og kovarians

① Marginalene

X, Y simultant fordelte

$f(x,y)$: sannsynlighets-tetthet for X og Y

X stokastisk variabel

og

Y stokastisk variabel

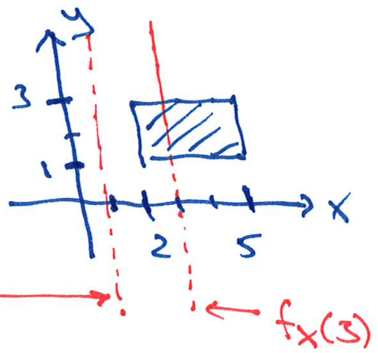
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

den marginale sannsynlighetstettheten til X

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

den marginale sannsynlighetstettheten til Y

Eks: $f(x,y) = \begin{cases} \frac{1}{91}xy^2 &, 2 \le x \le 5, 1 \le y \le 3 \\ 0 &, \text{ellers} \end{cases}$



$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ $\qquad f_X(1) = 0$

$2 \le x \le 5$: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_1^3 \frac{1}{91}xy^2\,dy = \frac{1}{91}x\left[\frac{1}{3}y^3\right]_1^3$

$= \frac{1}{91}x\left[\frac{1}{3}\cdot 3^3 - \frac{1}{3}\cdot 1^3\right) = \frac{x}{91\cdot 3}\cdot 26 = \frac{26}{\underbrace{91\cdot 3}_{7\cdot 13}}x$

$= \frac{2}{21}x$

Konklusjon: $f_X(x) = \begin{cases} \frac{2}{21}x, & 2 \le x \le 5 \\ 0, & \text{ellers} \end{cases}$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_2^5 x \cdot \left(\frac{2}{21}x\right)dx = \frac{2}{21}\left[\frac{1}{3}x^3\right]_2^5$$

$$= \frac{2}{21} \cdot \frac{1}{3} \cdot (5^3 - 2^3) = \frac{2}{21 \cdot 3}(125-8) = \frac{2 \cdot \cancel{117}^{39}}{21 \cdot \cancel{3}}$$

$$= \frac{78}{21} = \frac{26}{7}$$

Forklaring:

X, Y simultant fordelte

$f(x,y)$: tetthetsfunksjon $\longrightarrow$ $P(X \le a, Y \le b) = F(a,b)$

$$= \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)\,dy\,dx$$

X stokastisk variabel

$$P(X \le a) = F_X(a)$$

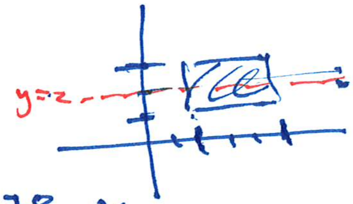
$$\| $$

$$P(X \le a, Y \text{ vilkårlig})$$

$$\|$$

$$\int_{-\infty}^{a} f_X(x)\,dx = \int_{-\infty}^{a}\underbrace{\int_{-\infty}^{\infty} f(x,y)\,dy}_{f_X(x)}\,dx$$

Eks:
$$f(x,y) = \begin{cases} \frac{1}{91} xy^2, & 2 \le x \le 5,\ 1 \le y \le 3 \\ 0, & \text{ellers} \end{cases}$$



$$f_X(x) = \begin{cases} \frac{2}{21} x, & 2 \le x \le 5 \\ 0, & \text{ellers} \end{cases}$$

$E(X) = \frac{78}{21} = \frac{26}{7}$

$\mathrm{Var}(X) = 69/98$

$$f_Y(y) = \begin{cases} \frac{3}{26} y^2, & 1 \le y \le 3 \\ 0, & \text{ellers} \end{cases}$$

$E(Y) = 30/13$

$\mathrm{Var}(Y) = 219/845$

$1 \le y \le 3$:
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_2^5 \frac{1}{91} xy^2\,dx = \frac{1}{91} y^2 \left[\frac{1}{2}x^2\right]_2^5 = \frac{y^2}{91 \cdot 2}(21)$$
$$= \frac{\cancel{21}^{\,3} \cdot y^2}{13 \cdot \cancel{7} \cdot 2} = \frac{3}{26} y^2$$

(2) Forventning, varians, kovarians når X, Y simultant fordelt w/ tetthetsfn. $f(x,y)$

Resultat:
Hvis $\underbrace{h(X,Y)}_{= Z}$ er en funksjon av X og Y, så er
$$\underbrace{E[h(X,Y)]}_{E[Z]} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) \cdot f(x,y)\,dy\,dx$$

Eks: $h(X,Y) = X$
$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f(x,y)\,dy\,dx = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

Eks: $h = XY$ $\quad E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot f(x,y)\, dy\, dx$

Vi fortsetter utregningene i eks:

$E(X) = 26/7$

$Var(X) = E(X^2) - E(X)^2 = \frac{203}{14} - \left(\frac{26}{7}\right)^2 = \frac{203 \cdot 7 - 2 \cdot 26^2}{2 \cdot 7 \cdot 7} = \frac{69}{98}$

$E(X^2) = \int_2^5 x^2 \cdot \frac{2}{21} x \, dx = \frac{2}{21}\left[\frac{1}{4}x^4\right]_2^5$

$= \frac{\cancel{2}}{21} \cdot \frac{1}{4} \cdot (625 - 16) = \frac{\cancel{2} \cdot \cancel{609}\ 203}{7\,\cancel{21} \cdot \cancel{4}\, 2} = \frac{203}{14}$

Kovarians: X, Y simultant fordelte med $\mu_X = E(X)$, $\mu_Y = E(Y)$

Defn: $Cov(X,Y) = E[(X - \mu_X)(Y - \mu_Y)]$

$= E(XY) - E(X) \cdot E(Y)$

I eks: $Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$

$= \frac{60}{7} - \frac{\cancel{26}\,2}{7} \cdot \frac{30}{\cancel{13}} = \frac{60}{7} - \frac{60}{7} = 0$

$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy = \int_1^3 y \cdot \frac{3}{26} y^2 \, dy = \frac{3}{26}\left[\frac{1}{4}y^4\right]_1^3$

$= \frac{3}{26} \cdot \frac{1}{4} \cdot (81 - 1) = \frac{3 \cdot \cancel{80}\,\cancel{20}\,10}{\cancel{4} \cdot \cancel{26}\,13} = \frac{30}{13}$

$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot f(x,y)\, dy\, dx = \int_2^5 \int_1^3 xy \cdot \frac{1}{91} xy^2 \, dy\, dx$

$= \int_2^5 \frac{x^2}{91} \left[\frac{1}{4}y^4\right]_1^3 dx = \int_2^5 x^2/91 \cdot 1/4 \cdot (81-1)\, dx$

$= \frac{1}{91} \cdot \frac{1}{\cancel{4}} \cdot \cancel{80}\,20 \left[\frac{1}{3}x^3\right]_2^5 = \frac{20}{91 \cdot 3}(125 - 8) = \frac{20 \cdot \cancel{117}\,\cancel{39}\,3}{\cancel{91} \cdot \cancel{3}\,\cancel{13} \cdot 7} = \frac{60}{7}$

Noen nyttige formler, X, Y simultant fordelt

i) $E(aX+bY+c) = aE(X) + bE(Y) + c$

ii) $Var(X) = E(X^2) - E(X)^2$ (tilsv. for Y)

iii) $Var(aX+bY+c) = a^2 Var(X) + 2ab\, Cov(X,Y) + b^2 Var(Y)$

iv) $Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$

v) $Cov(Y,X) = Cov(X,Y)$

vi) $Cov(X,X) = Var(X)$

$$Var(aX+bY+c) = E[(aX+bY+c)^2] - E[aX+bY+c]^2$$

$$= E[a^2X^2 + 2abXY + \cancel{2acX} + b^2Y^2 + \cancel{2bcY} + \cancel{c^2}]$$

$$- (aE(X) + bE(Y) + c)^2 \leftarrow a^2E(X)^2 + 2abE(X)E(Y) + \cancel{2ac\,E(X)} + b^2E(Y)^2 + \cancel{2bc\,E(Y)} + \cancel{c^2}$$

$$= a^2E(X^2) - a^2E(X)^2 + 2ab\,E(XY) - 2ab\,E(X)E(Y) + b^2E(Y^2) - b^2E(Y)^2$$

$$= a^2 \cdot Var(X) + 2ab\,Cov(X,Y) + b^2 Var(Y)$$

$$\iint (ax+by+c)\cdot f(x,y)\,dy\,dx = \iint axf(x,y)\,dy\,dx \leftarrow aE(X)$$

$$+ \iint byf(x,y)\,dy\,dx \leftarrow bE(Y)$$

$$+ \iint c\,f(x,y)\,dy\,dx \leftarrow c$$

$E(aX+bY+c)$

$a = E(X)$, $b = E(Y)$

$$Cov(X,Y) = \cancel{\iint (x-a)(y-b)\,dy}$$

$$= E[(X-a)(Y-b)] = E[XY - aY - bX + ab]$$

$$= E(XY) + E(-aY) + E(-bX) + E(ab)$$

$$= E(XY) - aE(Y) - bE(X) + ab$$

$$= E(XY) - ab\, \cancel{- b\cdot a + ab} = E(XY) - E(X)\cdot E(Y)$$

Eks (fortsatt)

$$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = \frac{363}{65} - \left(\frac{30}{13}\right)^2 = \frac{13\cdot 363 - 5\cdot 900}{5\cdot 13\cdot 13} = 219/845$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \int_1^3 y^2\cdot\frac{3}{26}y^2\,dy$$

$$= \frac{3}{26}\left[\frac{1}{5}y^5\right]_1^3 = \frac{3}{26}\cdot\frac{1}{5}\left(3^5 - 1^5\right) =$$

$$= \frac{3\cdot 1}{26\cdot 5}\cdot(243-1) = \frac{3\cdot\cancel{242}^{121}}{\cancel{26}_{13}\cdot 5} = \frac{363}{65}$$