Plan

1 Uavhengighet og kovarians
2 Betingede sannsynlighetsfordelinger

To hendelser A, B er uavhengige $\Leftrightarrow$ $P(A \cap B) = P(A) \cdot P(B)$

① Uavhengighet

X, Y simultant fordelte stok. variabler

Defn: X og Y er uavhengige hvis $f(x,y) = f_X(x) \cdot f_Y(y)$

Eks: $f(x,y) = \begin{cases} \frac{1}{91} xy^2 & , 2 \le x \le 5, 1 \le y \le 3 \\ 0 & , \text{ellers} \end{cases}$

$\frac{2}{7 \cdot \cancel{21}} \cdot \frac{\cancel{3}}{\cancel{26}_{13}} \cdot xy^2 = \frac{1}{91} xy^2$

$f_X(x) = \begin{cases} \frac{2}{21} x & , 2 \le x \le 5 \\ 0 & , \text{ellers} \end{cases}$

$f_Y(y) = \begin{cases} \frac{3}{26} y^2 & , 1 \le y \le 3 \\ 0 & , \text{ellers} \end{cases}$

$f_X(x) \cdot f_Y(y) = \begin{cases} \frac{2}{21} x \cdot \frac{3}{26} y^2 & , 2 \le x \le 5, 1 \le y \le 3 \\ 0 & , \text{ellers} \end{cases}$

Ser at $f_X(x) \cdot f_Y(y) = f(x,y)$,
så X og Y er uavhengige

Resultat:
X, Y simultant fordelte stokastiske variabler
Hvis X og Y er uavhengige, så er $Cov(X,Y)$

X og Y uavh $\Downarrow$ $Cov(X,Y) = 0$

Forklaring: X og Y uavh. $\Rightarrow f(x,y) = f_X(x) \cdot f_Y(y)$

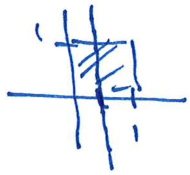
Da har vi $E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \cdot f(x,y)\, dy\, dx = \int \int xy \cdot f_X(x) \cdot f_Y(y)\, dy\, dx$

$= \int x f_X(x) \left(\int y f_Y(y)\, dy \right) dx = \int x f_X(x) E(Y)\, dx = E(Y) \cdot E(X)$

$\Rightarrow Cov(X,Y) = 0.$

Konsekvens: $\mathrm{Cov}(X,Y) \neq 0 \Rightarrow X, Y$ ikke uavhengige

Eks: $f(x,y) = \begin{cases} 6xy(2-x-y) & , 0 \le x,y \le 1 \\ 0 & , \text{ellers} \end{cases}$



$$\int_0^1\int_0^1 12xy - 6x^2y - 6xy^2 \, dy\,dx = 1 \leftarrow \text{Krav}$$

$$\int_0^1 \left[12x\cdot\tfrac{1}{2}y^2 - 6x^2\cdot\tfrac{1}{2}y^2 - 6x\cdot\tfrac{1}{3}y^3\right]_0^1 dx = \int_0^1 6x - 3x^2 - 2x \, dx$$

$$= \left[4\cdot\tfrac{1}{2}x^2 - 3\cdot\tfrac{1}{3}x^3\right]_0^1 = 2 - 1 = 1 \quad \text{ok}$$

Cov(X,Y): $\mathrm{Cov}(X,Y) = E(XY) - E(X)\cdot E(Y) = \cancel{?}\ \tfrac{1}{3} - \tfrac{7}{12}\cdot\tfrac{7}{12} = \frac{\cancel{?}\ 48 - 49}{144}$

$$E(XY) = \int_0^1\int_0^1 12x^2y^2 - 6x^3y^2 - 6x^2y^3 \, dy\,dx = \frac{-1}{144}$$

$$= \int_0^1 \left[12x^2\cdot\tfrac{1}{3}y^3 - 6x^3\cdot\tfrac{1}{3}y^3 - 6x^2\cdot\tfrac{1}{4}y^4\right]_0^1 dx = \int_0^1 \cancel{4x}\ 4x^2 - 2x^3 - \tfrac{3}{2}x^2 \, dx$$

$$\cancel{= \left[4\cdot\tfrac{1}{4}x^4 - \tfrac{3}{2}\cdot\tfrac{1}{3}x^3\right]_0^1 = 1 - \tfrac{1}{2} = \tfrac{1}{2}}$$

$$= \int_0^1 \tfrac{5}{2}x^2 - 2x^3 \, dx$$

$$= \left[\tfrac{5}{2}\cdot\tfrac{1}{3}x^3 - 2\cdot\tfrac{1}{4}x^4\right]_0^1 = \tfrac{5}{6} - \tfrac{2}{4} = \tfrac{1}{3}$$

$$E(X) = \int_0^1 x\cdot(4x - 3x^2)\,dx$$

$$= \int_0^1 4x^2 - 3x^3 \, dx = \left[4\cdot\tfrac{1}{3}x^3 - 3\cdot\tfrac{1}{4}x^4\right]_0^1$$

$$= \tfrac{4}{3} - \tfrac{3}{4} = \frac{16-9}{12} = \tfrac{7}{12}$$

$$f_X(x) = \int_0^1 f(x,y)\,dy$$

$$= \int_0^1 12xy - 6x^2y - 6xy^2 \, dy$$

$$= \left[12x\cdot\tfrac{1}{2}y^2 - 6x^2\cdot\tfrac{1}{2}y^2 - 6x\cdot\tfrac{1}{3}y^3\right]_0^1$$

$$= 6x - 3x^2 - 2x = 4x - 3x^2$$

$(0 \le x \le 1)$

$E(Y) = E(X)$ ved symmetri

$\mathrm{Cov}(X,Y) \neq 0 \Rightarrow X, Y$ ikke uavh.

$f_X(x) = \begin{cases} 4x - 3x^2 & , 0 \le x \le 1 \\ 0 & , \text{ellers} \end{cases}$ $\quad f_Y(y) = \begin{cases} 4y - 3y^2 & , 0 \le y \le 1 \\ 0 & , \text{ellers} \end{cases}$ (ved symmetri)

$f_X(x)\cdot f_Y(y) \neq f(x,y) \Rightarrow X, Y$ ikke uavh.

② Betingede sannsynlighetsfordelinger

Husk: A, B hendelser

$$p(A|B) = \frac{p(A\cap B)}{p(B)}$$

sannsynligheten for A, gitt B

Eks: Vi kaster en rød og blå terning

X = blå

Y = summen

$$p(Y=10 \mid X=5) = \frac{p(X=5, Y=10)}{p(X=5)}$$

$$= \frac{1/36}{1/6} = 1/6$$

$$p(Y=10) = 3/36$$

Det kontinuerlige tilfellet: X, Y kontinuerlige stokastiske variabler

X, Y simultant fordelte stok. var.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$f_{X|Y}(x|b) = \frac{f(x,b)}{f_Y(b)} = \frac{f(x,b)}{f_Y(b)}$$

betinget tetthetsfunksjon

Eks: $f(x,y) = \begin{cases} 6xy(2-x-y), & 0 \le x,y \le 1 \\ 0, & \text{ellers} \end{cases}$

$$f_Y(y) = \begin{cases} 4y-3y^2, & 0 \le y \le 1 \\ 0, & \text{ellers} \end{cases}$$

$b = 1/2$: $$f_{X|Y}(x|1/2) = \frac{f(x,1/2)}{f_Y(1/2)} = \frac{6x\cdot\frac{1}{2}(2-x-1/2)}{4\cdot 1/2 - 3\cdot(1/2)^2}$$

$$= \frac{3x(3/2-x)\cdot 4}{2-3/4 \quad \cdot 4} = \frac{6x(3-2x)}{5} = \underline{\frac{6}{5}x(3-2x)}$$

$$f_{X|Y}(x|1/2) = \begin{cases} \frac{6}{5}x(3-2x), & 0 \le x \le 1 \\ 0, & \text{ellers} \end{cases}$$

Dette er en tetthetsfn (oppfyller kravene)

i) $f_{X|Y}(x|1/2) \ge 0$ ok

ii) $\int_0^1 f_{X|Y}(x|1/2)\,dx = \int_0^1 \frac{6}{5}x(3-2x)\,dx = \int_0^1 \frac{18}{5}x - \frac{12}{5}x^2\,dx$

$= \left[\frac{18}{5}\cdot\frac{1}{2}x^2 - \frac{12}{5}\cdot\frac{1}{3}x^3\right]_0^1 = (9/5 - 4/5) - 0 = \underline{1}$ ok

Dette betyr at $\underbrace{X|Y=1/2} = X_{|Y=1/2} = X_{|1/2}$ $\frac{f(x,1/2)}{f_Y(1/2)}$ $\parallel$

Eks:

$E[\underbrace{X|Y=1/2}] = E[X_{|Y=1/2}] = \int_{-\infty}^{\infty} x\cdot f_{X|Y}(x|1/2)\,dx$

$= \int_0^1 x\cdot\frac{6}{5}x(3-2x)\,dx = \int_0^1 \frac{18}{5}x^2 - \frac{12}{5}x^3\,dx$

$= \left[\frac{18}{5}\cdot\frac{1}{3}x^3 - \frac{12}{5}\cdot\frac{1}{4}x^4\right]_0^1 = (6/5 - 3/5) - 0 = \underline{\underline{3/5}}$

Eks: X, Y stokastiske var. Simultant fordelt kont.

$$p(3 \le X \le 4 \mid Y = 2) = \frac{p(3 \le X \le 4, Y = 2)}{p(Y=2)}$$

gir ikke mening, $p(Y=2)=0$

istedet bruker vi h liten og:

$$p(3 \le X \le 4 \mid 2 \le Y \le 2+h) = \frac{p(3 \le X \le 4, 2 \le Y \le 2+h)}{p(2 \le Y \le 2+h)}$$

eller bedre:

$$\lim_{h \to 0} p(3 \le X \le 4 \mid 2 \le Y \le 2+h) = \lim_{h \to 0} \frac{p(3 \le X \le 4, 2 \le Y \le 2+h)}{p(2 \le Y \le 2+h)}$$

"0/0"

$$= \lim_{h \to 0} \frac{\int_3^4 \int_2^{2+h} f(x,y)\,dy\,dx}{\int_2^{2+h} f_Y(y)\,dy} = \lim_{h \to 0} \frac{\int_3^4 f(x, h+2) \cdot 1 \, dx}{f_Y(h+2) \cdot 1}$$

$$\boxed{\frac{d}{dx} \int_a^x f(t)\,dt = f(x)}$$

fundamental teoremet

$$= \frac{\int_3^4 f(x,2)\,dx}{f_Y(2)}$$

$$= \frac{1}{f_Y(2)} \int_3^4 f(x,2)\,dx$$

$$= \int_3^4 \frac{f(x,2)}{f_Y(2)}\,dx = \int_3^4 f_{X|Y}(x|2)\,dx$$

<u>Oppsummering:</u> 1) $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ tetthetsfn. for en stok. variabel $X|Y=y$

$$X|Y=y \;=\; X_{|Y=y} = X_{|y}$$

2) $\displaystyle\int_a^b f_{X|Y}(x|y)\,dx \;=\; \lim_{h\to 0} P(a \le X \le b \mid y \le Y \le y+h)$