Plan

1 Kovariansmatrisen

2 Anvendelser

① Kovariansmatrisen

X, Y stokastiske var.
a, b konstanter $\Big\}$ $Z = aX + bY$

$\mu_X = E(X)$
$\mu_Y = E(Y)$
$\underline{a} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$E(Z) = E(aX + bY) = E(aX) + E(bY)$$
$$= \underline{a \cdot E(X) + bE(Y)} = a\mu_X + b\mu_Y = (\mu_X \ \mu_Y) \cdot \underline{a}$$

$$Var(Z) = E[Z^2] - E[Z]^2 = E[(aX+bY)^2] - (aE(X)+bE(Y))^2$$
$$= E[a^2X^2 + 2abXY + b^2Y^2] - (a^2E(X)^2 + 2ab\,E(X)E(Y) + b^2E(Y)^2)$$
$$= a^2E(X^2) - a^2E(X)^2 + 2ab\,E(XY) - 2ab\,E(X)E(Y) + b^2E(Y^2) - b^2E(Y)^2$$
$$= \underline{a^2 Var(X) + 2ab\,Cov(X,Y) + b^2 Var(Y)}$$

Merk: $E(Z) = (\mu_X \ \mu_Y) \cdot \begin{pmatrix} a \\ b \end{pmatrix}$

$$Var(Z) = (a \ \ b) \cdot \begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= (a \ \ b) \underbrace{\begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{pmatrix}}_{\Sigma} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \underline{a}^T \Sigma \cdot \underline{a} \qquad \Sigma \text{ kovariansmatrisen}$$

Merk: (1) $\mathrm{Var}(Z) \geq 0 \Rightarrow \underline{a}^T \Sigma \underline{a} \geq 0$ for alle $\underline{a}$

Altså er Σ alltid positiv semidefin.

(2) $\mathrm{Cov}(X,Y)=0 \Rightarrow \mathrm{Var}(Z) = a^2 \cdot \mathrm{Var}(X) + b^2 \cdot \mathrm{Var}(Y)$

(3) Kan utvides til $n>2$ simultant fordelte stokastiske variabler

$X_1, X_2, \ldots, X_n$ simultant fordelte stok. var.

$a_1, a_2, \ldots, a_n$ konstanter $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

$\Downarrow$

$Z = a_1 X_1 + a_2 X_2 + \ldots + a_n X_n$

Da har vi:

$E(Z) = a_1 E(X_1) + a_2 E(X_2) + \ldots + a_n E(X_n)$

$= \left(E(X_1)\ E(X_2) \ldots E(X_n) \right) \cdot \underline{a}$ lin. form

$\mathrm{Var}(Z) = \underline{a}^T \Sigma \underline{a}$, der $a_1^2 \mathrm{Var}(X_1) + a_1 a_2 \mathrm{Cov}(X_1, X_2) + \ldots$

$$\Sigma = \begin{pmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1,X_2) & \ldots & \mathrm{Cov}(X_1,X_n) \\ \mathrm{Cov}(X_1,X_2) & \mathrm{Var}(X_2) & \ldots & \\ \vdots & \vdots & \ddots & \end{pmatrix}$$

kovariansmatrisen

$\mathrm{Var}(Z) \geq 0 \Rightarrow \Sigma$ positiv semidefin.

② Anvendelser : X_1, X_2, X_3 : avkastninger til tre verdipapirer

$\underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

$w_1 + w_2 + w_3 = 1$ (portefølje)

$Z = w_1 X_1 + w_2 X_2 + w_3 X_3$

avkastn. til porteføljen

$$E(Z) = \underbrace{\big(E(X_1) \;\; E(X_2) \;\; E(X_3)\big)}_{\mu} \cdot \underline{w}$$

$$Var(Z) = \underline{w}^T \Sigma \underline{w}$$

Eks:

$E(X_1) = 4$
$E(X_2) = 2$
$E(X_3) = 3$

$$\Sigma = \begin{pmatrix} Var(X_1) & Cov(X_1,X_2) & Cov(X_1,X_3) \\ \cdot & Var(X_2) & Cov(X_2,X_3) \\ \cdot & \cdot & Var(X_3) \end{pmatrix}$$

$$\underline{\mu} = (4 \;\; 2 \;\; 3) \qquad = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix}$$

$$\min \; \underset{\underline{a}^T \Sigma \underline{a}}{\overset{\shortparallel}{Var(Z)}} \quad \text{når} \quad \begin{cases} E(Z) = r \longleftarrow & \underline{\mu} \cdot \underline{w} = r \\ w_1 + w_2 + w_3 = 1 & \underset{\underline{1}}{\underset{\shortparallel}{(1\;1\;1)}} \cdot \underline{w} = 1 \end{cases}$$

$$L(\underline{w}; \lambda_1, \lambda_2) = \underline{w}^T \Sigma \underline{w} - \lambda_1(\underline{\mu} \cdot \underline{w} - r) - \lambda_2(\underline{1}\,\underline{w} - 1)$$

FOC: $\left.\begin{matrix} L'_{w_1} = 0 \\ L'_{w_2} = 0 \\ L'_{w_3} = 0 \end{matrix}\right\}$

$$2\Sigma \underline{w} - \lambda_1 (\underline{\mu}^T) - \lambda_2 (\underline{1}^T) = \underline{0}$$

$$2\Sigma \underline{w} = \lambda_1 \underline{\mu}^T + \lambda_2 \underline{1}^T \quad | :2$$

$$\Sigma \underline{w} = \frac{\lambda_1}{2} \underline{\mu}^T + \frac{\lambda_2}{2} \underline{1}^T$$

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix} \cdot \underline{w} = \frac{\lambda_1}{2} \cdot \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} + \frac{\lambda_2}{2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\underline{C}$: $\quad \mu \cdot \underline{w} = (4\ 2\ 3)\,\underline{w} = r$

$\quad\quad \underline{1} \cdot \underline{w} = (1\ 1\ 1)\,\underline{w} = 1$

$$\Sigma = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix} \qquad \underline{\text{Kofaktorene}}: \begin{pmatrix} 14 & -8 & -4 \\ -8 & 6 & 2 \\ -4 & 2 & 2 \end{pmatrix}$$

$$|\Sigma| = 2\cdot 14 + 2\cdot(-8) + 2\cdot(-4) = 4 \neq 0$$

$$\Sigma^{-1} = \frac{1}{4}\cdot\begin{pmatrix} 14 & -8 & -4 \\ -8 & 6 & 2 \\ -4 & 2 & 2 \end{pmatrix} = \underline{\frac{1}{2}\begin{pmatrix} 7 & -4 & -2 \\ -4 & 3 & 1 \\ -2 & 1 & 1 \end{pmatrix}}$$

$\underline{\text{FOC}}$:
$$\Sigma\cdot\underline{w} = \frac{\lambda_1}{2}\begin{pmatrix}4\\2\\3\end{pmatrix} + \frac{\lambda_2}{2}\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \Big|\ \Sigma^{-1}\cdot$$

$$\underline{w} = \Sigma^{-1}\left(\frac{\lambda_1}{2}\right)\begin{pmatrix}4\\2\\3\end{pmatrix} + \Sigma^{-1}\left(\frac{\lambda_2}{2}\right)\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \frac{\lambda_1}{2}\cdot\underbrace{\frac{1}{2}\begin{pmatrix} 7 & -4 & -2 \\ -4 & 3 & 1 \\ -2 & 1 & 1 \end{pmatrix}}_{\Sigma^{-1}}\begin{pmatrix}4\\2\\3\end{pmatrix} + \frac{\lambda_2}{2}\,\frac{1}{2}\begin{pmatrix} 7 & -4 & -2 \\ -4 & 3 & 1 \\ -2 & 1 & 1 \end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\underline{w} = \frac{\lambda_1}{4}\cdot\begin{pmatrix}14\\-7\\-3\end{pmatrix} + \frac{\lambda_2}{4}\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$\underline{C1}$:
$$(4\ 2\ 3)\cdot\left(\frac{\lambda_1}{4}\begin{pmatrix}14\\-7\\-3\end{pmatrix} + \frac{\lambda_2}{4}\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \frac{\lambda_1}{4}\,(33) + \frac{\lambda_2}{4}\,(4)$$

$$= \frac{33}{4}\lambda_1 + \lambda_2 = r$$

$\underline{C2}$:
$$(1\ 1\ 1)\cdot\left(\left(\frac{\lambda_1}{4}\right)\cdot\begin{pmatrix}14\\-7\\-3\end{pmatrix} + \left(\frac{\lambda_2}{4}\right)\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \frac{\lambda_1}{4}\cdot 4 + \frac{\lambda_2}{4}\cdot 1$$

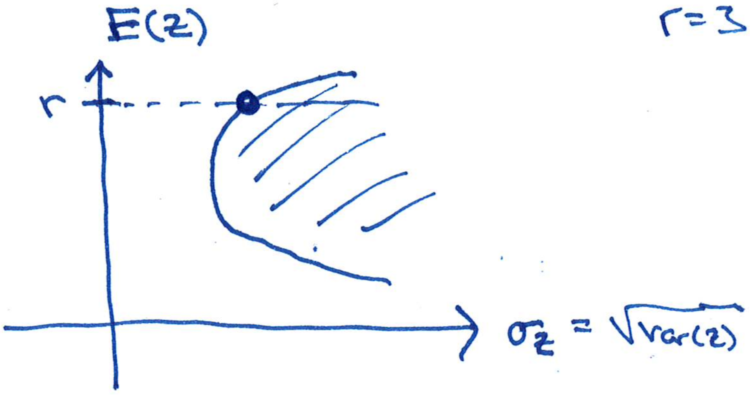
$$= \lambda_1 + \frac{1}{4}\lambda_2 = 1 \quad\Rightarrow\quad \lambda_1 = 1 - \frac{1}{4}\lambda_2$$

$$\frac{33}{4}\left(1 - \frac{1}{4}\lambda_2\right) + \lambda_2 = r$$

$$33/4 - 33/16\,\lambda_2 + \lambda_2 = r \quad\Rightarrow\quad -\frac{17}{16}\lambda_2 = r - 33/4 \quad\Rightarrow\quad \lambda_2 = \underline{\frac{16}{17}\left(33/4 - r\right)}$$

-- regner og regner -- $\Rightarrow \underline{w} = \frac{1}{17}\begin{pmatrix} 10r-23 \\ 28-7r \\ 12-3r \end{pmatrix}$

$r=3: \quad \underline{w} = \frac{1}{17}\begin{pmatrix} 7 \\ 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 7/17 \\ 7/17 \\ 3/17 \end{pmatrix}$



Gitt denne problemstillingen: $\min \underset{\underline{w}^T \Sigma\, \underline{w}}{\underset{\shortparallel}{Var(Z)}}$ når $\begin{cases} \underline{\mu}\cdot\underline{w} = r \\ \underline{1}\cdot\underline{w} = 1 \end{cases}$

FOC: $2\Sigma\cdot\underline{w} - \lambda_1\cdot\underline{\mu}^T - \lambda_2\cdot\underline{1}^T = \underline{0}$

$\Sigma\,\underline{w} = \frac{\lambda_1}{2}\underline{\mu}^T + \frac{\lambda_2}{2}\underline{1}^T$

$\underline{w} = \frac{\lambda_1}{2}\Sigma^{-1}\underline{\mu}^T + \frac{\lambda_2}{2}\Sigma^{-1}\underline{1}^T$

Anta: $|\Sigma| \neq 0$

$\Uparrow$

Σ positiv defin.

C: $\underline{\mu}\cdot\underline{w} = r \Rightarrow \frac{\lambda_1}{2}\underline{\mu}\Sigma^{-1}\underline{\mu}^T + \frac{\lambda_2}{2}\underline{\mu}^T\Sigma^{-1}\underline{1}^T = r$

$\frac{1}{2}(\underline{\mu}\Sigma^{-1}\underline{\mu}^T)\lambda_1 + \frac{1}{2}(\underline{\mu}\Sigma^{-1}\underline{1}^T)\lambda_2 = r$

$\underline{1}\cdot\underline{w} = 1 \qquad \frac{1}{2}(\underline{1}\Sigma^{-1}\underline{\mu}^T)\lambda_1 + \frac{1}{2}(\underline{1}\cdot\Sigma^{-1}\underline{1}^T)\lambda_2 = 1$

Anta: ① Σ^{-1} positiv defin.
② $\underline{\mu}, \underline{1}$ lineært uavh.

Merk: Σ pos. defin. $\Rightarrow \Sigma^{-1}$ pos. defin.

Eks $\underline{\mu} = (4\ 2\ 3)$ $\quad \Sigma = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix}$

ønsker å fjerne betingelsen $\omega_1 + \omega_2 + \omega_3 = 1$

Legger til en bank-konto:

$$\underline{\mu}^* = (0\ \boxed{4\ \ 2\ \ 3}) \qquad \Sigma^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 3 & 1 \\ 0 & 2 & 1 & 5 \end{pmatrix}$$

$\uparrow$ bank-kto.

$$\underline{w} = \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} \qquad E(w_0 B + w_1 X_1 + w_2 X_2 + w_3 X_3) = \underline{\mu} \cdot \underline{w}$$

$$Var(\ --\ \ \text{''}\ \ --\ ) = \underline{w}^T \Sigma \underline{w}$$

$$w_0 = 1 - (w_1 + w_2 + w_3)$$

$$\min \underset{\underline{w}^T \Sigma \underline{w}}{\overset{\text{''}}{Var(Z)}} \quad \text{når} \quad \underset{\underline{\mu} \cdot \underline{w} = r}{\overset{\text{''}}{E(Z) = r}}$$

FOC: $2\Sigma \underline{w} = \lambda \cdot \underline{\mu}^T$

$\Sigma \underline{w} = \frac{\lambda}{2} \underline{\mu}^T$

$\underline{w} = \frac{\lambda}{2} \Sigma^{-1} \underline{\mu}^T$

C: $\frac{\lambda}{2} (\underline{\mu} \Sigma^{-1} \underline{\mu}^T) = r$

$$\lambda = \frac{2r}{\underline{\mu} \Sigma^{-1} \underline{\mu}^T}$$

$$\underline{w} = \frac{r}{\underline{\mu} \Sigma^{-1} \underline{\mu}^T} \cdot \Sigma^{-1} \underline{\mu}^T \quad \longleftarrow$$

Eks: $(4\ 2\ 3) \cdot \frac{1}{2} \begin{pmatrix} 7 & -4 & -2 \\ -4 & 3 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{2} (14\ -7\ -3) \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{2}(33) = \frac{33}{2}$

$$\frac{1}{2} \begin{pmatrix} 7 & -4 & -2 \\ -4 & 3 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 14 \\ -7 \\ -3 \end{pmatrix}$$

$$\Rightarrow \underline{w} = \frac{r}{33/2} \cdot \frac{1}{2} \begin{pmatrix} 14 \\ -7 \\ -3 \end{pmatrix} = \frac{r}{33} \begin{pmatrix} 14 \\ -7 \\ -3 \end{pmatrix}$$

r=3: $w_1 = 14/11$ "long"
$w_2 = -7/11$ "short"
$w_3 = -3/11$ "short"
$w_0 = -4/11$ "borrow"