Plan

1 Introduksjon til differensial-likninger
2 Separable differensial-likninger

[E] (deler av) 7.1–7.6 + 7.9–7.10
[DA] Oppgaver + oppsummering

① Intro til diff. likninger

Defn: En differensial-likning med ukjent $y = y(t)$ er en likning som forbinder $y'(t)$, og evt. høyere ordens deriverte.

Eks:

$y' = te^t$ $\qquad y'(t) = t\cdot e^t$

$y' = ty$ $\qquad y'(t) = t\cdot y(t)$

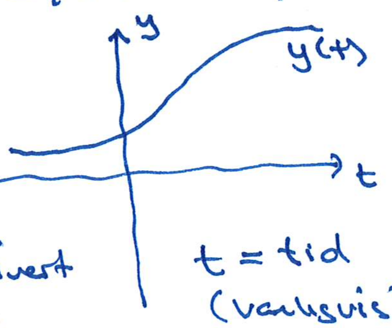
$y' = ty^2$ $\qquad \vdots$

$y' = t^2\cdot y$

$y''\cdot y = y'$

En løsning er en funksjon $y = y(t)$ som passer i diff. likningen.

Den generelle løsningen: alle løsninger av diff. likningen

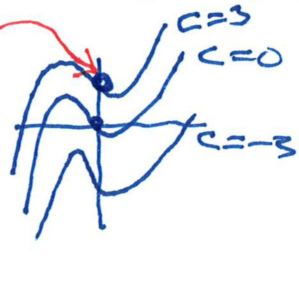


t = tid (vanligvis)

Ex: $y' = t^2 - t$ ← y er en antiderivert til fn. $t^2 - t$

$$y = \int t^2 - t \, dt = \tfrac{1}{3}t^3 - \tfrac{1}{2}t^2 + C$$

Generell løsn: $y = \frac{1}{3}t^3 - \frac{1}{2}t^2 + C$



En spesiell / partikulær løsn w/ $y(0) = 3$ initial betingelse $\qquad y = \frac{1}{3}t^3 - \frac{1}{2}t^2 + 3$

Orden: Høyeste orden av den deriverte som inngår.

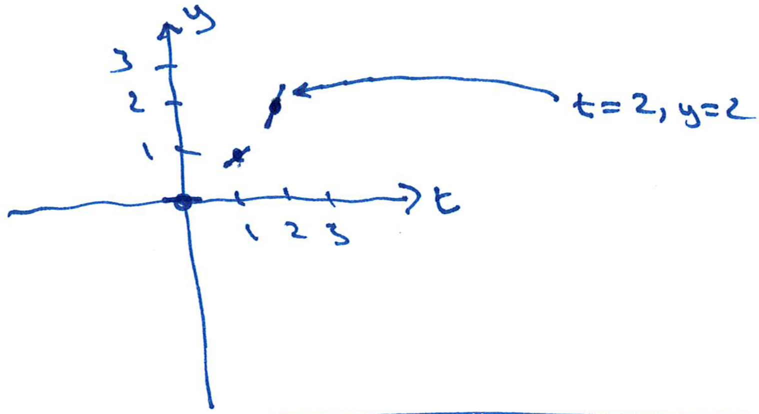
Eks: $y' = yt$ første ordens diff. likn.

$y'' - y = t$ andre ordens diff. likn.

Første ordens diff. likninger

Kan (ofte) skrives på formen: $y' = F(t,y)$

Fasediagram:



Resultat:
En første ordens diff.likning $y' = F(t,y)$ har generell løsning som avhenger av *en* ubestemt konstant C.

Eks: $y' = yt \quad |:y$

$\frac{y'}{y} = t$

$\frac{1}{y}\cdot y' = t \quad | \int \ldots dt$

$\int \frac{1}{y}\cdot \underbrace{y'\,dt}_{dy} = \int t\,dt$

$\boxed{\begin{array}{l} y = y(t) \\ dy = y'\,dt \end{array}}$

~~$\int y'\,dt = \int y \cdot t\,dt$ ($\int y'\,dt = y$) ikke mulig å løse~~

$\int \frac{1}{y}\,dy = \int t\,dt$

$\ln|y| + C_1 = \frac{1}{2}t^2 + C_2$ implisitt løsning

$$\ln|y| = \frac{1}{2}t^2 + c_2 - c_1 \quad | \; e^{\cdot}$$

$$e^{\ln|y|} = e^{\frac{1}{2}t^2 + c_2 - c_1}$$

$$|y| = e^{\frac{1}{2}t^2} \cdot e^{c_2 - c_1}$$

$$y = \underline{\pm e^{c_2-c_1} \cdot e^{\frac{1}{2}t^2}} \qquad y = \underline{\underline{C \cdot e^{\frac{1}{2}t^2}}} \quad \left(C = \pm e^{c_2 - c_1}\right)$$

generell løsn.

$y' = ty$, $y(0) = 1/2$ $\quad (t=0, y=1/2)$

$$y = C \cdot e^{\frac{1}{2}t^2}$$

$$\frac{1}{2} = C \cdot e^{\frac{1}{2}\cdot 0^2} = C \cdot 1 = C$$

$$\Rightarrow C = 1/2$$

Part. løsn: $y = \underline{\underline{\frac{1}{2}e^{\frac{1}{2}t^2}}}$

(2) Separable diff. likninger

Defn: En første ordens diff. likning kalles separabel hvis den kan skrives

$$y' = f(y) \cdot g(t)$$

Metode:

$$y' = f(y) \cdot g(t) \quad | : f(y)$$

$$\frac{1}{f(y)} y' = g(t) \quad | \int \ldots dt$$

$$\int \frac{1}{f(y)} \underbrace{y' dt}_{dy} = \int g(t)\, dt$$

$$\int \frac{1}{f(y)} dy = \int g(t) dt$$

implisitt løsn

eksplisitt løsn.

$$\Rightarrow F(y) + C_1 = G(t) + C_2$$

$$F(y) = G(t) + C \qquad (C = C_2 - C_1)$$

$$y' = \frac{dy}{dt}$$

$$\frac{1}{f(y)} \frac{dy}{dt} = g(t)$$

$$\int \frac{1}{f(y)} dy = \int g(t) dt$$

Eks: $y' = y + t$ ← kan ikke faktorisere $y + t = f(t) \cdot g(y)$

$\Rightarrow$ ikke separabel

$\cancel{y' - y = t}$

$\cancel{\int y' - y \, dt = \int t \, dt}$

Eks: $\boxed{y' = ry}$ (r konstant) $r > 0$ ← eksponensiell vekst

$y' = y \cdot r \quad | : y$

$\frac{1}{y} y' = r$

$\int \frac{1}{y} dy = \int \frac{1}{y} y' dt = \int r \, dt$

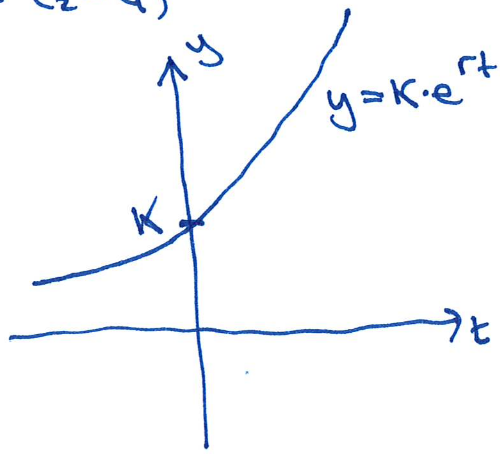
$\ln|y| = rt + C \qquad (C = C_2 - C_1)$

$|y| = e^{rt} \cdot e^{C} = e^{rt + C}$

$y = \pm e^{C} \cdot e^{rt}$

$y = K \cdot e^{rt}$

$y(0) = K \cdot e^{0} = K$



Eks: $y' = e^{y+t} = e^{y} \cdot e^{t}$

$\frac{1}{e^y} y' = e^t$

$\int e^{-y} dy = \int \frac{1}{e^y} y' dt = \int e^t \, dt$

$\boxed{\begin{aligned} u &= -y \\ du &= -1 \cdot dy \end{aligned}}$

$\int e^u \frac{du}{-1} = \int -e^u \, du = -e^u + C$

$-e^{-y} = e^t + C \quad | \cdot (-1)$

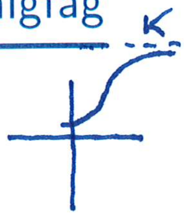
$e^{-y} = -e^t - C \quad | \ln(\cdot)$

$-y = \ln(-e^t - C) \quad | \cdot (-1)$

$y = -\ln(-e^t - C) = \ln\left(\frac{1}{-e^t - C}\right)$

$e^t < -C$

$t < \ln(-C) = \ln(-C)$



Eks: $y' = ry(1 - y/K)$ $r = 1/5$ $K = 100$

logistisk diff.likn

separabel

$$y' = \underbrace{\tfrac{1}{5}}_{g(t)}\underbrace{y\cdot(1 - y/100)}_{f(y)}$$

$$\frac{100\cdot 1}{100\cdot y(1 - y/100)}\, y' = \frac{1}{5}$$

$$\frac{100}{y\cdot(100-y)} = \frac{\text{“}A\text{”}}{y} + \frac{\text{“}B\text{”}}{100-y}$$

$$100 = A\cdot(100-y) + By$$

$$= 100A - Ay + By$$

$$100 = \underset{0}{\underset{\shortparallel}{(B-A)}}\,y + \underset{100}{\underset{\shortparallel}{(100A)}}$$

$B = 1$ $A = 1$

$$\int \frac{100}{y\cdot(100-y)}\,\cancel{y'\,dt}\;dy = \int \frac{1}{5}\,dt$$

$$\int \frac{1}{y} + \frac{1}{100-y}\,dy = \frac{1}{5}t + C$$

$$\ln|y| - \ln|100-y| = \frac{1}{5}t + C$$

$$\ln\frac{|y|}{|100-y|} = \frac{1}{5}t + C$$

$$\left|\frac{y}{100-y}\right| = e^{\frac{1}{5}t + C} = e^{\frac{1}{5}t}\cdot e^{C}$$

$$\frac{y}{100-y} = \pm e^{C} e^{\frac{1}{5}t} = Ke^{\frac{1}{5}t} \qquad |\cdot(100-y)$$

$$y = (100-y)\cdot Ke^{\frac{1}{5}t}$$

$$y + y\cdot Ke^{\frac{1}{5}t} = 100K\cdot e^{\frac{1}{5}t}$$

$$y = \frac{100K e^{\frac{1}{5}t}}{1 + Ke^{\frac{1}{5}t}} = 100\cdot\frac{Ke^{\frac{1}{5}t}}{1 + Ke^{\frac{1}{5}t}}$$