Plan
1 Superposisjonsprinsippet
2 Tilfellet med konstante koeffisienter

Første ordens lineær diff.likn: $y' + a(t)\cdot y = b(t)$

Andre ordens — " — — : $y'' + a(t)y' + b(t)y = c(t)$

1 Superposisjonsprinsippet

$$\boxed{y = y_h + y_p}$$

Eks: $\boxed{y' + 3y = t^2}$

$$\frac{d}{dt}y + 3\cdot y = t^2$$

$$\left(\frac{d}{dt}+3\right)\cdot y = t^2$$

$$D(y) = t^2$$

$D = \frac{d}{dt} + 3$ differensial-operator

$$D(t) = \frac{d}{dt}(t) + 3\cdot t$$
$$= 1 + 3t$$

$$\text{Funksjon} \xrightarrow{D} \text{Funksjon}$$

$t \longrightarrow 1+3t$

$t^2 \longrightarrow 2t + 3t^2$

$\vdots$

$1 \longrightarrow 3\cdot 1$

Int. faktor: $u = e^{\int 3dt} = e^{3t}$

$$(y\cdot e^{3t})' = t^2 e^{3t}$$

$$ye^{3t} = \int t^2 e^{3t}\,dt$$

$$ye^{3t} = \frac{1}{3}t^2e^{3t} - \frac{2}{9}te^{3t} + \frac{2}{27}e^{3t} + C$$

$$y = \underline{\underline{\frac{1}{3}t^2 - \frac{2}{9}t + \frac{2}{27} + C\cdot e^{-3t}}}$$

$$D\left(\frac{1}{3}t^2 - \frac{2}{9}t + \frac{2}{27}\right)$$

$$D(t^2) = 2t + 3t^2$$

$$\frac{1}{3}D(t^2) = \frac{2}{3}t + t^2$$

$$\frac{1}{3}D(t^2) - \frac{2}{9}D(t) = t^2 + \cancel{\frac{2}{3}t} - \frac{2}{9}(1 + \cancel{3t})$$

$$= t^2 - \frac{2}{9}$$

$$= \frac{1}{3}D(t^2) - \frac{2}{9}D(t) + \frac{2}{27}D(1) = t^2$$

$$\int t^2 e^{3t}\,dt = \frac{1}{3}e^{3t}\cdot t^2 - \int \frac{1}{3}e^{3t}\cdot 2t\,dt$$

$$\boxed{\begin{array}{ll} u = \frac{1}{3}e^{3t} & v = t^2 \\ u' = e^{3t} & v' = 2t \end{array}}$$

$$= \frac{1}{3}t^2 e^{3t} - \frac{2}{3}\int t\cdot e^{3t}\,dt =$$

$$\boxed{\begin{array}{ll} u = \frac{1}{3}e^{3t} & v = t \\ u' = e^{3t} & v' = 1 \end{array}}$$

$$= \frac{1}{3}t^2 e^{3t} - \frac{2}{3}\left(\frac{1}{3}e^{3t}\cdot t - \int \frac{1}{3}e^{3t}\cdot 1\,dt\right)$$

$$= \underline{\frac{1}{3}t^2 e^{3t} - \frac{2}{9}te^{3t} + \frac{2}{9}\cdot\frac{1}{3}e^{3t} + C}$$

Merk:

Defn En operator D er lineær hvis følgende betingelser holder:

i) $D(y_1+y_2) = D(y_1) + D(y_2)$ — y_1, y_2 funksjoner

ii) $D(a\cdot y) = a\cdot D(y)$ — y funksjon, a et tall

Resultat:

① Hvis $D = \frac{d}{dt} + a(t)$ så er D lineær

$y' + a(t)\cdot y = b(t)$
$D(y) = b(t)$

② Hvis $D = \frac{d^2}{dt^2} + a(t)\cdot\frac{d}{dt} + b(t)$ så er D lineær

$y'' + a(t)y' + b(t)y = c(t)$
$D(y) = c(t)$

"Differensialoperatorer for lineære diff. lik. er lineære operatorer".

Eks: $y' + 3y = t^2 \rightsquigarrow D = \frac{d}{dt} + 3$.

er lineær:

$$D(y_1+y_2) = (y_1+y_2)' + 3(y_1+y_2)$$
$$= y_1' + y_2' + 3y_1 + 3y_2$$
$$= (y_1' + 3y_1) + (y_2' + 3y_2)$$
$$= D(y_1) + D(y_2)$$

Superposisjonsprinsippet

Gjelder for alle lineære diff.likninger (uansett orden)

$$D(y) = h(t)$$

$y = y_h + y_p$

"Den generelle løsn. av diff. likn. y er summen av den homogene løsn y_h og en partikulær løsn. y_p"

y_h = generell løsn. av den homogene diff. likn. $D(y) = 0$

y_p = en spesiell løsn. av $D(y) = h(t)$

Eks: $y' + 3y = t^2$ lineær, første orden

$D(y) = t^2$ der $D = \frac{d}{dt} + 3$

y_h: $y' + 3y = 0$ eller $D(y) = 0$

$y' = -3y \Rightarrow y_h = C \cdot e^{-3t}$ $\qquad D(y_h) = 0$

y_p: $y' + 3y = t^2$

$$y_p = \frac{1}{3}t^2 - \frac{2}{9}t + \frac{2}{27} \qquad D(y_p) = t^2$$

$$y = y_h + y_p = C \cdot e^{-3t} + \frac{1}{3}t^2 - \frac{2}{9}t + \frac{2}{27}$$

1 $D(y_h + y_p) = D(y_h) + D(y_p) = 0 + t^2 = t^2$

(2) Hvis y er en løsn. av diff.likn.
dvs $D(y) = t^2$
Da er
$$D(y - y_p) = D(y) - D(y_p)$$
$$= t^2 - t^2 = 0$$
$$\Downarrow$$
$$y - y_p = y_h$$
$$y = y_h + y_p$$

(2) Tilfellet med konstante koeffisienter

$y' + ay = b(t)$ første ordens lineær diff.likning m/ konstante koeff.

Eks: $y' - 4y = 1$

Superpos.-prinsippet:
$$y = y_h + y_p$$
$$= \underline{\underline{C \cdot e^{4t} - 1/4}}$$

y_h: $y' - 4y = 0$

Karakteristisk likn: $r - 4 = 0$
$r = 4 \rightsquigarrow y_h = \underline{C \cdot e^{4t}}$

$y = e^{rt}$
$y' = e^{rt} \cdot r$
Setter inn:
$(r \cdot e^{rt}) - 4(e^{rt}) = 0$
$(r - 4) \cdot e^{rt} = 0$
$\underline{r - 4 = 0}$

y_p: $y' - 4y = \underline{\underline{1}}$: $y = A$ konst. løsn., $y' = 0$ } Setter inn: $0 - 4(A) = 1$ $A = -1/4$ $y_p = \underline{-1/4}$

$$\int a(t)\,dt = \int 2\,dt = 2t + C \Rightarrow u = e^{2t}$$

Alt: $y' + 2y = t \quad | \cdot e^{2t}$

$$(y \cdot e^{2t})' = te^{2t}$$

$$y \cdot e^{2t} = \int t \cdot e^{2t}\,dt$$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}e^{2t} & v = t \\ u' = e^{2t} & v' = 1 \end{array}}$$

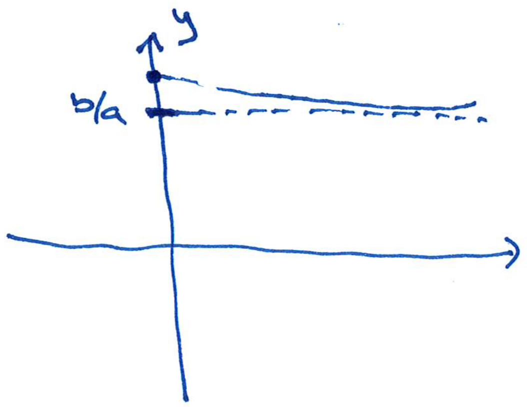
$$= \frac{1}{2}e^{2t} \cdot t - \int \frac{1}{2}e^{2t} \cdot 1\,dt$$

$$ye^{2t} = \frac{1}{2}te^{2t} - \frac{1}{2} \cdot \frac{1}{2}e^{2t} + C$$

$$ye^{2t} = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C$$

$$y = \underbrace{\frac{1}{2}t - 1/4}_{y_p} + \underbrace{c \cdot e^{-2t}}_{y_h}$$

Eks: $y' + ay = b \Rightarrow y = C \cdot e^{-at} + \frac{b}{a}$



$$y(0) = C \cdot e^{0} + \frac{b}{a} = C + b/a$$

$$y_0 = C + b/a \Rightarrow C = y_0 - b/a$$

$y_0 = b/a$: $C = 0$

$y_0 > b/a$: $C > 0$

Eks: $y' + ay = b$ (a, b konst.)

$$y = y_h + y_p = \underline{\underline{C \cdot e^{-at} + b/a}}$$

y_h: $y' + ay = 0$

$r + a = 0 \quad r = -a \Rightarrow y_h = \underline{C \cdot e^{-at}}$

y_p: $y' + ay = b \quad y = A$ (konst.)

$y' = 0$

$\Downarrow$

$0 + a(A) = b$

$A = b/a \Rightarrow y_p = \underline{b/a}$

Eks: $y' + 2y = t$

$$y = y_h + y_p = \underline{\underline{C \cdot e^{-2t} + \tfrac{1}{2}t - 1/4}}$$

y_h: $y' + 2y = 0 \quad r + 2 = 0$

$r = -2 \quad \Rightarrow \quad y_h = \underline{C \cdot e^{-2t}}$

y_p: $y' + 2y = \underline{\underline{t}} \quad y = At + B \Rightarrow y_p = \underline{+\tfrac{1}{2}t - 1/4}$

$y' = A$

$\Downarrow$

$(A) + 2(A\underline{t} + B) = t$

$\underline{2A \cdot t} + \underline{(A + 2B)} = \underline{t}$

$2A = 1 \quad A + 2B = 0$

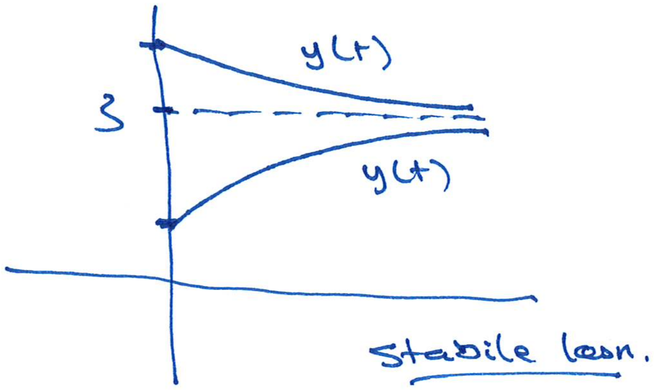
$A = \underline{1/2} \quad \tfrac{1}{2} + 2B = 0$

$2B = -1/2$

$\underline{B = -1/4}$

$$y' + 2y = 6 \qquad y = C \cdot e^{-2t} + 3$$

$$y(0) = C + 3 \Rightarrow C = \underline{y_0 - 3}$$



$y_0 = 3$: $\quad y = \underline{3}$

$y_0 > 3$: $\quad C > 0$

$y_0 < 3$: $\quad C < 0$