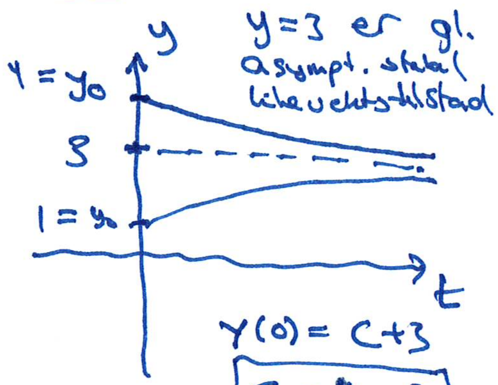
Plan
1 Likevektstilstander og stabilitet
2 Oppgaveregning: Differensiallikninger

① Likevektstilstander og stabilitet

Eks: $y' + 4y = 12$ lineær, første ordens



$$y = y_h + y_p = \underline{\underline{C \cdot e^{-4t} + 3}}$$

$y(0) = C + 3$

$\boxed{C = y_0 - 3}$

y_h: $y' + 4y = 0$

$r + 4 = 0$

$\underline{r = -4} \rightarrow y_h = C \cdot \underline{e^{-4t}}$

y_p: $y' + 4y = 12$ konst.

Konst løsn: $\left.\begin{aligned} y &= A \\ y' &= 0 \end{aligned}\right\}$ $0 + 4A = 12$

$\underline{A = 3}$

$y_p = \underline{3}$

(equilibrium state)

Defn: En <u>likevektstilstand</u> er en konstant løsning $y = y_e$ (dvs $y' = 0$, og evt $y'' = 0$).

Eks: $y = 3$ er likevektstilstand ($y_e = 3$)

Defn: En likevektstilstand $y = y_e$ er stabil hvis y_0 nært $y_e \Rightarrow y(t) \to y_e$ når $t \to \infty$, ellers er den ustabil. En stabil likevektstilstand $y = y_e$ er globalt asymptotisk stabil hvis $y(t) \to y_e$ når $t \to \infty$ for alle verdier av C / y_0.

Ex:

$$y'' = 3 - 4y' - 3y$$

$$y'' + 4y' + 3y = 3$$ Lineær andreordens konst. koeff.

$y_e = 1$ likevektstilstand

$$y = y_h + y_p = C_1 \cdot e^{-3t} + C_2 e^{-t} + 1$$ glob.as. stabil likevekt

y_h: $y'' + 4y' + 3y = 0$

$$r^2 + 4r + 3 = 0$$

$$(r+3)(r+1) = 0$$

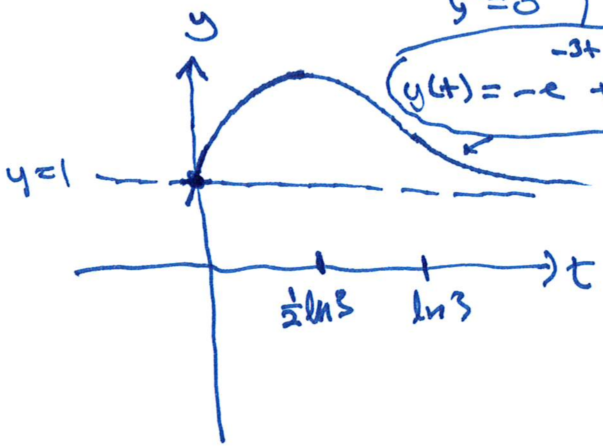
$$r = -3, \quad r = -1 \quad \to \quad y_h = C_1 \cdot e^{-3t} + C_2 e^{-t}$$

y_p: $y'' + 4y' + 3y = 3$

$$\left.\begin{array}{l} y = A \\ y' = 0 \\ y'' = 0 \end{array}\right\} \quad 0 + 4 \cdot 0 + 3A = 3$$

$$A = 1 \qquad y_p = 1$$

$$y(t) = -e^{-3t} + e^{-t} + 1$$



Eks: $y'' = 3 - 4y' - 3y$, $y(0) = 1$, $y'(0) = 2$

$y(0) = 1$: $C_1 e^0 + C_2 \cdot e^0 + 1 = 1$ $\quad C_1 + C_2 = 0$

$y'(0) = 2$: $-3C_1 \cdot e^0 - C_2 e^0 + 0 = 2$ $\quad -3C_1 - C_2 = 2$

$$-2C_1 = 2$$

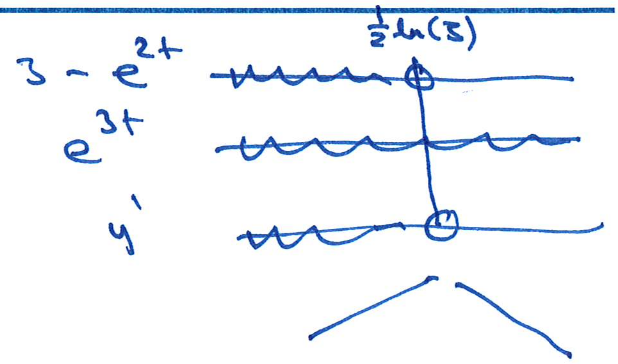
$$C_1 = -1$$

$$C_2 = 1$$

⇒ part. løsn.

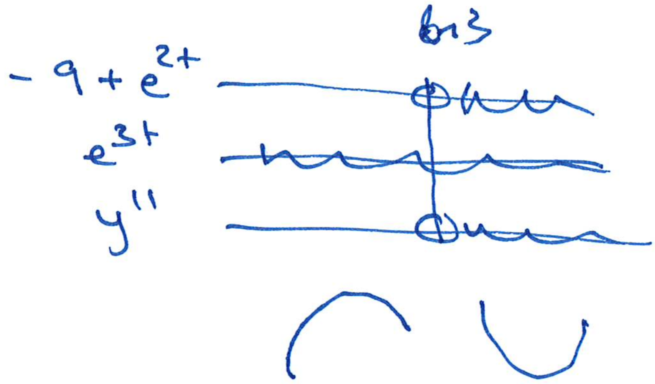
$$y(t) = -e^{-3t} + e^{-t} + 1$$

$$y' = 3e^{-3t} - e^{-t}$$
$$= \frac{3 - e^{2t}}{e^{3t}} = 0$$



$$3 - e^{2t} = 0$$
$$e^{2t} = 3$$
$$2t = \ln(3)$$
$$t = \tfrac{1}{2}\ln(3)$$

$$y'' = -9e^{-3t} + e^{-t}$$
$$= \frac{-9 + e^{2t}}{e^{3t}} = 0$$



$$e^{2t} = 9$$
$$2t = \ln 9 = 2\ln 3$$
$$t = \ln 3$$

Når bruker vi likevektstilstander / stabilitet?

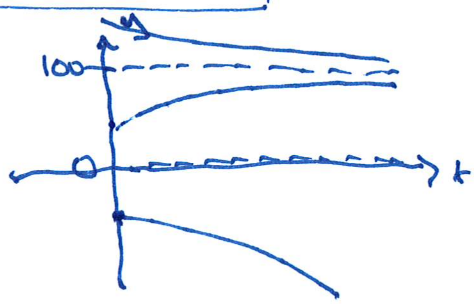
- For alle diff.likninger som er autonome, dvs at t ikke opptrer eksplisitt i diff.likningen.
- Særlig nyttig hvis det er vanskelig å løse diff.likn. eksplisitt.

Eks: ~~$y' = \frac{1}{5}y(100-y)$~~ $\boxed{y' = \frac{1}{5}y(1 - y/100)}$

Likevektstilstand:

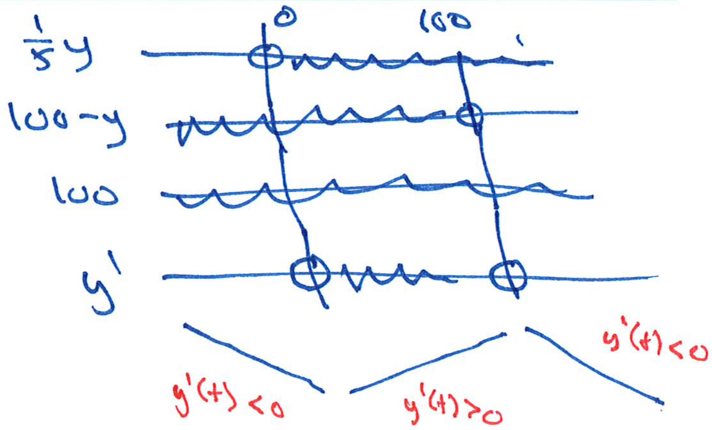
$y' = 0$: $\frac{1}{5}y \cdot (1 - y/100) = 0$

$y = 0$ eller $y/100 = 1$
$y = 100$



Likevektstilstander: $y_e = 0$, $y_e = 100$
ustabil stabil

$$y' = \frac{1}{5}y\,(1 - y/100)$$
$$= \frac{1}{5}y\left(\frac{100-y}{100}\right)$$



Eksplisitt løsn:

$$y' = \frac{1}{5}y\,\frac{100-y}{100}$$

$$\frac{100}{100-y}\,y' = \frac{1}{5}y$$

$$\frac{100}{y(100-y)}\,y' = \frac{1}{5}$$

$$\int \frac{1}{y} + \frac{1}{100-y}\,dy = \int \frac{1}{5}\,dt = \frac{1}{5}t + C$$

$$\ln|y| - \ln|100-y| = \frac{1}{5}t + C$$

$$\ln\left|\frac{y}{100-y}\right| = \frac{1}{5}t + C$$

$$\frac{y}{100-y} = \pm\, e^{\frac{1}{5}t + C} = K\cdot e^{\frac{1}{5}t}$$

$$y = K\cdot e^{\frac{1}{5}t}(100-y)$$

$$y\cdot(1 + K\cdot e^{\frac{1}{5}t}) = 100\cdot K e^{\frac{1}{5}t}$$

$$y = 100\,\frac{K e^{\frac{1}{5}t}}{1 + K\cdot e^{\frac{1}{5}t}}$$ generell løsn.

Husk:

$$y(0) = 100\cdot\frac{K e^0}{1 + K\cdot e^0}$$

$$y_0 = \frac{100K}{1+K}$$

($\Rightarrow$ asymptote hvis $-1<K<0$ $\Leftrightarrow$ $y_0<0$)

$t\to\infty$: $e^{\frac{1}{5}t} \to \infty$

$$\frac{K e^{\frac{1}{5}t}}{1 + K e^{\frac{1}{5}t}} \to 1$$

$$y(t) \to 100$$

Stabil likevektstilstand!

(2) Repetisjon / Oppgaver: Diff. lign.

Første ordens separable

$y' = f(y)\cdot g(t)$

Metode: $\frac{1}{f(y)}\, y' = g(t)$

$$\int \frac{1}{f(y)}\, dy = \int g(t)\, dt$$

— " — lineære

$y' + a(t)\,y = b(t)$

Metode: Integrerende faktor

$u = e^{\int a(t)\,dt}$

$\downarrow$

$(y\cdot u)' = b(t)\cdot u$

$y\cdot u = \int b(t)\cdot u\, dt$

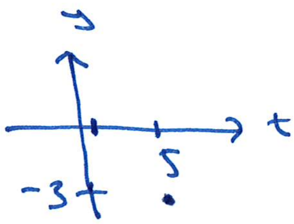
Andre ordens lineære (m/ konstante koeff.)

$y'' + ay' + by = h(t)$

Metode: Superposisjon

$y = y_h + y_p$

Oppgave 13.8.1 [DA]:



$y\cdot y' = 1, \quad y(5) = -3$

$y' = \frac{1}{y}$

$y\cdot y' = 1$

$\int y\, dy = \int 1\, dt$

$\frac{1}{2}y^2 = t + C$

$\rightarrow y^2 = 2(t+C) = 2t + 2C$

$y = \pm\sqrt{2t+K} \quad (K = 2C)$

$y(5) = -3: \quad -3 = \pm\sqrt{2\cdot 5 + K}$

$-3 = -\sqrt{10+K}$

$9 = 10 + K \qquad K = -1$

$y = -\sqrt{2t-1} \quad (t \geq 1/2)$

Ex: $y'' - \frac{4}{25}y' - \frac{16}{1250}y = 0$, $y(0)=1$, $y(25)=e^4$

linear homogen:

$$r^2 - \frac{2}{25}r - \frac{16}{1250} = 0$$

$$r = \frac{\frac{2}{25} \pm \sqrt{\left(\frac{2}{25}\right)^2 - 4\left(-\frac{16}{1250}\right)}}{2}$$

$$= \frac{1}{25} \pm \frac{1}{2}\sqrt{\frac{4}{625} + \frac{\cancel{64}\;32}{\cancel{1250}\;625}}$$

$$= \frac{1}{25} \pm \frac{1}{2}\sqrt{\frac{6^2}{25^2}} = \frac{1}{25} \pm \frac{1}{2}\cdot\frac{6}{25} = \frac{1 \pm 3}{25}$$

$$= \frac{2}{50} \pm \frac{6\cdot 5}{10\cdot 5} = \frac{32}{50}, \frac{-28}{50} = \frac{16}{25}, -\frac{14}{25}$$

$$y = C_1 \cdot e^{\frac{16}{25}t} + C_2 \cdot e^{-\frac{14}{25}t} = \frac{e^{18}-1}{e^{30}-1}e^{\frac{16}{25}t} + \frac{e^{30}-e^{18}}{e^{30}-1}e^{-\frac{14}{25}t}$$

$y(0)=1$: $C_1 \cdot e^0 + C_2 \cdot e^0 = 1$ $\qquad C_1 + C_2 = 1$

$y(25)=e^4$: $C_1 \cdot e^{16} + C_2 \cdot e^{-14} = e^4$ $\qquad C_1 e^{16} + C_2 e^{-14} = e^4$

$C_2 = 1 - C_1$: $C_1 \cdot e^{16} + (1 - C_1)e^{-14} = e^4$

$$C_1 \cdot e^{16} - C_1 e^{-14} = e^4 - e^{-14}$$

$$\frac{C_1\left(e^{16} - e^{-14}\right)}{e^{16} - e^{-14}} = \frac{e^4 - e^{-14}}{e^{16}\cdot e^{-14}}$$

$$C_1 = \frac{e^4 - e^{-14}}{e^{16} - e^{-14}}\cdot\frac{e^{14}}{e^{14}} = \frac{e^{18}-1}{e^{30}-1}$$

$$C_2 = 1 - \frac{e^{18}-1}{e^{30}-1} = \frac{(e^{30}-\cancel{1}) - (e^{18}-\cancel{1})}{e^{30}-1}$$

$$C_2 = \frac{e^{30}-e^{18}}{e^{30}-1}$$

$$y = C_1 \cdot e^{\frac{4}{25}t} + C_2 \cdot e^{-\frac{2}{25}t}$$

$y(0) = 1: \quad C_1 + C_2 = 1 \qquad C_2 = 1 - C_1$

$y(25) = e^4: \quad C_1 \cdot e^4 + C_2 e^{-2} = e^4$

$$C_1 \cdot e^4 + (1 - C_1) e^{-2} = e^4$$

$$C_1 \cdot (e^4 - e^{-2}) = e^4 - e^{-2}$$

$$C_1 = \frac{e^4 - e^{-2}}{e^4 - e^{-2}} = \underline{1} \qquad C_2 = \underline{0}$$

$$y = e^{\frac{4}{25}t}$$