Plan

1 Variasjonsregning
2 Eulers likning

Ingen foreles. i morgen, men både tirs/ons neste uke.

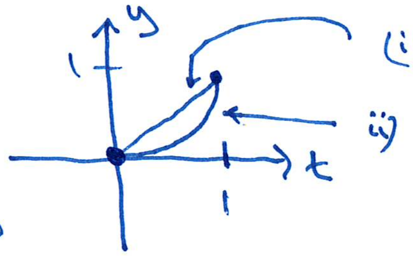
1 Variasjonsregning

Funksjonal: $J(y)$ $\quad y = y(t)$ funksjon, $J(y)$ et tall

Eks: $J(y) = \int_0^1 4ty - (y')^2 \, dt$ for $y(0) = 0$, $y(1) = 1$

i) $y(t) = t$: $y'(t) = 1$

$$J(y) = \int_0^1 4t \cdot t - (1)^2 \, dt = \int_0^1 4t^2 - 1 \, dt = \left[\tfrac{4}{3}t^3 - t\right]_0^1 = (4/3 - 1) - 0 = \underline{\underline{1/3}}$$

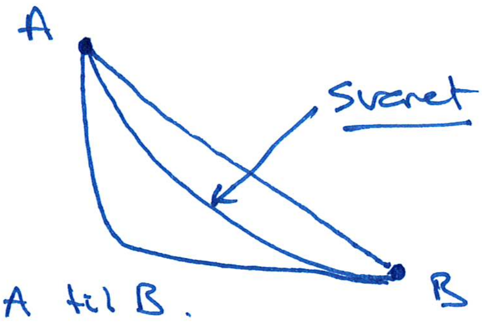


ii) $y(t) = t^2$: $y'(t) = 2t$

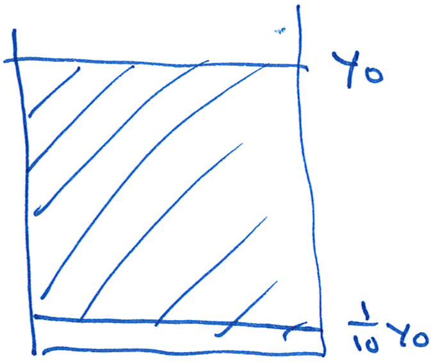
$$J(y) = \int_0^1 4t \cdot t^2 - (2t)^2 \, dt = \int_0^1 4t^3 - 4t^2 \, dt = \left[t^4 - \tfrac{4}{3}t^3\right]_0^1 = (1 - 4/3) - 0 = \underline{\underline{-1/3}}$$

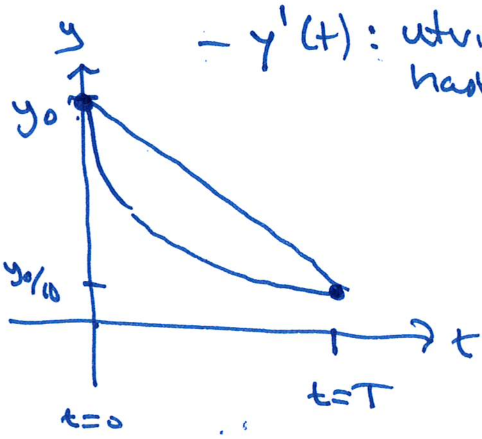
$$\boxed{\textbf{Variasjonsproblem:} \quad \max_{y=y(t)}/\min \underbrace{\int_a^b F(t, y, y') \, dt}_{J(y)} \quad \text{når} \quad \underbrace{\begin{cases} y(a) = y_0 \\ y(b) = y_1 \end{cases}}_{\text{bibetingelser}}}$$

Eks: Anta at A ligger høyere enn B, og ikke rett under hverandre. Hvilken kurve gir minst mulig tid fra A til B.



Eks:





$-y'(t)$: utvinnings hastighet

olje reservoar

$y(t)$: mengde olje ved tiden t

$t=0: \; y(0) = Y_0$

$t=T: \; y(T) = \frac{1}{10} Y_0$

$$\max \int_0^T F(t,y,y')\, e^{-rt}\, dt$$

når $\begin{cases} y(0) = Y_0 \\ y(T) = \frac{1}{10} Y_0 \end{cases}$

$F(t,y,y') =$
$p(t) \cdot y - C(t,y,y')$

Eks: max $\int_0^1 \underbrace{4ty - (y')^2}_{F(t,y,y')}\,dt$ når $\begin{cases} y(0)=0 \\ y(1)=1 \end{cases}$

(2) Kandidatpunkt: Euler-likningen (Euler-Lagrange)

$$F'_y - \frac{d}{dt}\left(F'_{y'}\right) = 0$$

Eks: $F = 4ty - (y')^2$

$$F'_y = 4t$$

$$F'_{y'} = -2y' \Rightarrow \frac{d}{dt}\left(F'_{y'}\right) = \frac{d}{dt}\left(-2y'\right) = -2y''(t)$$

Euler: $4t - (-2y'') = 0$

$$4t + 2y'' = 0$$

$$2y'' = -4t$$

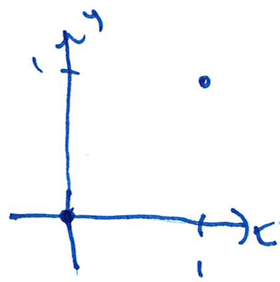
$$y'' = -2t \leftarrow \boxed{y'' + ay' + by = h(t)}$$

$$y' = \int -2t\,dt$$

$$y' = -t^2 + C$$

$$y = \int -t^2 + C\,dt$$

$$y = \underbrace{-\frac{1}{3}t^3}_{y_p} + \underbrace{Ct + D}_{y_h}$$



$y(0)=0$

$y(1)=1$

$y = y_h + y_p$

$= C_1 + C_2 t - \frac{1}{3}t^3$

y_h: $r^2 = 0$

$r_1 = r_2 = 0$

$y_h = C_1 \cdot e^{0t} + C_2 t \cdot e^{0t}$

$= C_1 + C_2 t$

y_p: $\cancel{y = At^2}$, $\cancel{y' = 2At}$, $\cancel{y'' = 2A}$

$y = At^3$, $y' = 3At^2$, $y'' = 6At$

$6At = -2t$

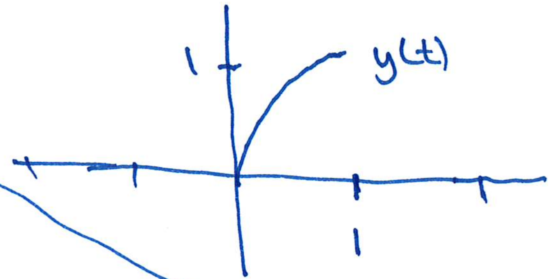
$6A = -2$ $\quad A = -2/6 = -1/3$ $\quad y_p = -\frac{1}{3}t^3$

$$y = -\frac{1}{3}t^3 + Ct + D$$

$y(0)=0$: $\quad 0 = -0 + 0 + D \Rightarrow \underline{D=0}$

$y(1)=1$: $\quad 1 = -\frac{1}{3}\cdot 1^3 + C\cdot 1 + \underset{\overset{\shortparallel}{0}}{D} \Rightarrow C = 1 + 1/3 = \underline{4/3}$

Kandidat: $\underline{y(t) = -\frac{1}{3}t^3 + 4/3\, t} = \frac{1}{3}(4t - t^3) = \frac{t}{3}(4-t^2)$



Teori:
Hvis $y^*(t)$ er maks/min i et variasjonsproblem

$$\max/\min \int_a^b F(t,y,y')\,dt \quad \text{når} \quad \begin{cases} y(a) = y_0 \\ y(b) = y_1 \end{cases}$$

så oppfyller den Euler-likningen $F'_y - \frac{d}{dt}(F'_{y'}) = 0$.

Eks: Kandidat for max: $y^*(t) = \underline{\frac{t}{3}(4-t^2)} = \underline{\frac{4}{3}t - \frac{1}{3}t^3}$

$$J(y^*(t)) = \int_0^1 4ty - (y')^2\,dt$$

$$= \int_0^1 4t\cdot\left(\frac{4}{3}t - \frac{1}{3}t^3\right) - \left(\frac{4}{3} - t^2\right)^2 dt$$

$$= \int_0^1 \frac{16}{3}t^2 - \frac{4}{3}t^4 - \frac{16}{9} + \frac{8}{3}t^2 - t^4\,dt$$

$$= \left[\frac{8}{3}t^3 - \frac{7}{3}\cdot\frac{1}{5}t^5 - \frac{16}{9}t\right]_0^1 = \frac{8}{3} - \frac{7}{15} - \frac{16}{9}$$

$$= \frac{8\cdot 15 - 7\cdot 3 - 16\cdot 5}{45} = \frac{120 - 21 - 80}{45} = \underline{\underline{19/45}} \simeq 0.40$$

③ Tilstrekkelig betingelse for max/min

$$F = F(t,y,y') : \begin{pmatrix} F''_{yy} & F''_{yy'} \\ F''_{y'y} & F''_{y'y'} \end{pmatrix}$$

Betingelser:

i) $\left. \begin{array}{l} (F''_{yy})\cdot(F''_{y'y'}) - (F''_{yy'})^2 \geq 0 \\ F''_{yy} \, , \, F''_{y'y'} \geq 0 \\ \text{(F er konveks i } (y,y')\text{)} \end{array} \right\} \Longrightarrow$ alle kandidatfn. $y^*(t)$ som oppfyller Euler-likn. + ~~start~~-betingelsene er **min**

ii) $\left. \begin{array}{l} (F''_{yy})\cdot(F''_{y'y'}) - (F''_{yy'})^2 \geq 0 \\ F''_{yy} \, , \, F''_{y'y'} \leq 0 \\ \text{(F er konkav i } (y,y')\text{)} \end{array} \right\} \Longrightarrow$ alle kand. $y^*(t)$ som oppfyller Euler-likn. + startbet. er **max**

Eks: $F = 4ty - (y')^2$

$$F'_y = \underline{4t} \qquad \begin{pmatrix} F''_{yy} & F''_{yy'} \\ F''_{y'y} & F''_{y'y'} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$$

$$F'_{y'} = \underline{-2y'}$$

F er konkav i (y,y') $\Longleftarrow$ $\begin{cases} \det = 0\cdot(-2) - 0^2 = 0 \ \checkmark \\ 0, -2 \leq 0 \ \checkmark \end{cases}$

$\Downarrow$

$y^*(t) = \underline{\frac{4}{3}t - \frac{1}{3}t^3}$ er **max** med max-verdi $19/45 \simeq \underline{0.42}$

Forklaring: konvekse / konkave funksjoner

Eks: $f(x,y) = 7x - 4y + y^2 - 4xy + 5x^2$
$= 5x^2 - 4xy + y^2 + 7x - 4y$

$f'_x = \underline{10x - 4y + 7} = 0$ $\quad 10x - 4y = -7$
$f'_y = \underline{-4x + 2y - 4} = 0$ $\quad -4x + 2y = 4 \;\;]\, 2$
$2x = 1 \quad \boxed{x = 1/2,\; y = 3}$ stasj. pkt.

$$H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 10 & -4 \\ -4 & 2 \end{pmatrix}$$

$(x^*, y^*) = \underline{(1/2, 3)}$

$$H(f)(1/2,3) = \begin{pmatrix} f''_{xx}(1/2,3) & f''_{xy}(1/2,3) \\ f''_{xy}(1/2,3) & f''_{yy}(1/2,3) \end{pmatrix} = \begin{pmatrix} 10 & -4 \\ -4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$\left.\begin{array}{l} \det = AC - B^2 > 0 \\ A, C > 0 \end{array}\right\}$ lok. min.

Defn: f konveks $\Leftrightarrow$ $H(f)(x,y)$ pos. semidefn. for alle (x,y)
f konkav $\Leftrightarrow$ $H(f)(x,y)$ neg. — " — — " —

Husk: A symm 2×2 matrise
λ_1, λ_2 egenverdier

A pos. defn. $\Leftrightarrow$ $\lambda_1, \lambda_2 > 0$ $\Leftrightarrow$ $|A| = \lambda_1 \cdot \lambda_2 > 0$, $tr(A) = \lambda_1 + \lambda_2 > 0$

pos. semidefn. $\Leftrightarrow$ $\lambda_1, \lambda_2 \geq 0$
$\Updownarrow$
$|A| = \lambda_1 \cdot \lambda_2 \geq 0$
$tr(A) = \lambda_1 + \lambda_2 \geq 0$

Neste forelesning: Gjennomgang av Opps 6 fra eksamen V2013:

$$\boxed{\max/\min \int_0^3 \ln(4y - y')\,dt \quad \text{når} \quad \begin{array}{l} y(0) = 3 \\ y(3) = -9e^{12} \end{array}}$$