Plan

1 Optimal kontrollteori
2 Pontryagins maksimumsprinsipp

Eksamen 04/2013

max / min $\int_0^3 \underbrace{\ln(4y-y')}_{F(t,y,y')} dt$ når $\begin{cases} y(0)=3 \\ y(3)=-9e^{12} \end{cases}$

(a) Euler-likningen: $F'_y - \frac{d}{dt}(F'_{y'}) = 0$

$F = \ln(4y-y') = \ln(u)$, $u = 4y-y'$

$F'_y = \frac{1}{u} \cdot u'_y = \frac{4}{u}$

$F'_{y'} = \frac{1}{u} \cdot u'_{y'} = \frac{-1}{u} \Rightarrow \frac{d}{dt} F'_{y'} = \frac{d}{dt}(-1 \cdot u^{-1}) = -1 \cdot (-1) u^{-2} \frac{du}{dt}$

$= \frac{1}{u^2} \cdot (4y' - y'')$

Euler: $\frac{4}{u} - \frac{1}{u^2}(4y'-y'') = 0 \quad | \cdot u^2$

$4u - (4y'-y'') = 0$

$4(4y-y') - (4y'-y'') = 0$

$y'' - 8y' + 16y = 0$, $y(0)=3$, $y(3) = -9e^{12}$

$y = y_h + \underset{0}{\underset{\shortparallel}{y_p}} = (C_1 + C_2 t)e^{4t}$

$y(0)=3$: $3 = (C_1 + \cancel{C_2 \cdot 0})e^{4 \cdot 0}$

$C_1 = 3$

$y(3) = -9e^{12}$: $-9\cancel{e^{12}} = (3+3C_2)\cancel{e^{12}}$

$-9 = 3 + 3C_2$

$-12 = 3C_2 \quad C_2 = -4$

y_h: $r^2 - 8r + 16 = 0$

$(r-4)^2 = 0$

$r_1 = r_2 = 4 \rightarrow \boxed{(C_1 + C_2 t)e^{4t} = y_h}$

Løsn. av Euler-Likn. som oppfyller bibetingelsene:

$$\underline{y^* = (3-4t)e^{4t}} \quad \leftarrow \text{(eneste) kandidat for max/min}$$

(b) <u>F konveks/konkav?</u> $F = \ln(u), \quad u = 4y - y'$

$$F'_y = \frac{4}{u} = 4 \cdot u^{-1} \qquad F''_{yy} = 4 \cdot (-1) u^{-2} \cdot u'_y = -16 u^{-2} = \underline{-\frac{16}{u^2}}$$

$$F'_{y'} = \frac{-1}{u} = -1 \cdot u^{-1} \qquad F''_{yy'} = 4(-1)u^{-2} \cdot u'_{y'} = 4u^{-2} = \underline{\frac{4}{u^2}}$$

$$H(F) = \begin{pmatrix} \overset{A}{-\frac{16}{u^2}} & \overset{B}{\frac{4}{u^2}} \\ \underset{B}{\frac{4}{u^2}} & \underset{C}{-\frac{1}{u^2}} \end{pmatrix} \qquad \left.\begin{array}{l} F''_{y'y} \\ F''_{y'y'} \end{array}\right\} = (-1) \cdot (-1) u^{-2} \cdot u'_{y'} = \underline{-\frac{1}{u^2}}$$

$$\det H(F) = AC - B^2 = (-16/u^2)(-1/u^2) - (4/u^2)^2 = \frac{16}{u^4} - \frac{16}{u^4} = 0$$

$$\operatorname{tr} H(F) = A + C = -17/u^2 < 0$$

$\Downarrow$

F <u>konkav</u> i (y, y')

$\Downarrow$

$$\underline{\underline{y^* = (3-4t)e^{4t}}} \quad \text{gir } \underline{\underline{\text{max}}}$$

Husk:

i) $\det H(F) \geq 0$, $\operatorname{tr} H(F) \geq 0$ $\Downarrow$ F konveks

ii) $\det H(F) \geq 0$, $\operatorname{tr} H(F) \leq 0$ $\Downarrow$ F konkav

<u>max-verdi:</u> $\max/\min \int_0^3 \ln(4y - y')\,dt$ når $y(0) = 3$, $y(3) = -9e^{12}$

$$\int_0^3 \ln(4y - y')\,dt$$

$$= \int_0^3 \ln(4e^{4t})\,dt$$

$$= \int_0^3 \ln(4) + 4t\,dt = \left[\ln(4)\cdot t + 2t^2\right]_0^3$$

$$= (3\ln(4) + 18) - 0 = \underline{3\ln(4) + 18} \simeq 22$$

$$\left\{\begin{array}{l} y = (3-4t)e^{4t} \\ y' = -4e^{4t} + (3-4t)e^{4t} \cdot 4 \\ \quad = (8-16t)e^{4t} \\ 4y - y' = 4(3-\cancel{4t})e^{4t} - (8 - \cancel{16t})e^{4t} \\ \quad = 4e^{4t} \end{array}\right.$$

Merk: Hvis F er en kvadratisk fcn i (y, y') og den symmetriske matrisen til A har egenverdier λ_1, λ_2, så har vi

$$F \text{ konveks} \iff \underset{\text{(pos. semidef.)}}{\lambda_1, \lambda_2 \geq 0} \iff \begin{matrix} \det A = \lambda_1 \cdot \lambda_2 \geq 0 \\ \operatorname{tr} A = \lambda_1 + \lambda_2 \geq 0 \end{matrix}$$

$$F \text{ konkav} \iff \underset{\text{(neg. semidef.)}}{\lambda_1, \lambda_2 \leq 0} \iff \begin{matrix} \det A = \lambda_1 \cdot \lambda_2 \geq 0 \\ \operatorname{tr} A = \lambda_1 + \lambda_2 \leq 0 \end{matrix}$$

1 Optimal kontroll teori

max/min $\int_a^b F(t,y,u)\,dt$ når $\begin{cases} y' = G(t,y,u) \\ y(a) = y_0 \\ y(b) = y_1 \end{cases}$

$y(t)$: tilstandsvariabel

$u(t)$: kontroll variabel

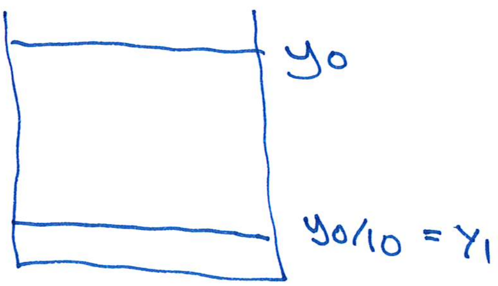
Spesialtilfelle: $G(t,y,u) = u$: $y' = u$

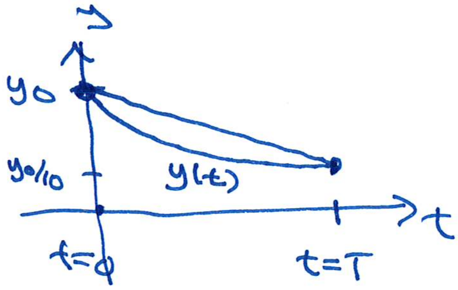
$\Downarrow$

Variasjons-problem: max/min $\int_a^b F(t,y,y')\,dt$ når $\begin{cases} y(a) = y_0 \\ y(b) = y_1 \end{cases}$

$-y'$: utvinningshastighet

Eks:





oljereservoar : $y(t)$ = mengde olje i reservoaret

Eks: $\max \int_0^1 y - u^2 \, dt$ når $\begin{cases} y' = y + u \\ y(0) = 1 \\ y(1) = e - \frac{1}{2e} + \frac{1}{2} \end{cases}$

$$\begin{cases} F = y - u^2 \\ G = y + u \end{cases}$$

① **Skriv om til variasjonsproblem:**

$$y' = y + u \Rightarrow u = y' - y$$

$$\max \int_0^1 y - (y'-y)^2 \, dt \quad \text{når} \quad \begin{cases} y(0) = 1 \\ y(1) = e - \frac{1}{2e} + \frac{1}{2} \end{cases}$$

$$F = y - (y'-y)^2$$

$$F'_y = 1 - 2(y'-y)\cdot(-1) = 1 + 2(y'-y)$$

$$F'_{y'} = 0 - 2(y'-y)\cdot 1 = -2(y'-y)$$

$$\Rightarrow \frac{d}{dt} F'_{y'} = -2(y'' - y')$$

Euler: $1 + 2(y'-y) + 2(y''-y') = 0$

$$2y'' - 2y = -1 \quad |:2$$

$$\boxed{y'' - y = -1/2}$$

Kar. likn: $r^2 - 1 = 0 \quad r = \pm 1$

$$y = y_h + y_p = C_1 e^t + C_2 e^{-t} + 1/2$$

$y(0) = 1$: $C_1 + C_2 + 1/2 = 1$ $\boxed{C_1 + C_2 = 1/2}$

$y(1) = e - \frac{1}{2e} + \frac{1}{2}$: $C_1 \cdot e + C_2 \cdot \frac{1}{e} \cancel{+1/2} = e - \frac{1}{2e} \cancel{+1/2}$

$$C_1 = 1 \quad C_2 = -\frac{1}{2}$$

Konkl: $y^* = e^t - \frac{1}{2}e^{-t} + 1/2$ kand. funksjon

$F = y - (y'-y)^2$

$$\left.\begin{aligned} F'_y &= \underline{1 + 2(y'-y)} \\ F'_{y'} &= \underline{-2(y'-y)} \end{aligned}\right\} \quad H(F) = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \quad \begin{aligned} \det &= 4-4=0 \\ \mathrm{tr} &= -2-2=-4 \end{aligned}$$

F konkav $\Rightarrow y^* = e^t - \frac{1}{2}e^{-t} + \frac{1}{2}$ max

(2) Løse som optimal kontrollteori-problem

Eks: $\max \int_0^1 y - u^2\, dt$ når $\begin{cases} y' = y+u \\ y(0)=1 \\ y(1) = e - \frac{1}{2e} + 1/2 \end{cases}$

$$\boxed{\begin{aligned} F &= y - u^2 \\ G &= y+u \end{aligned}}$$

Hamiltonian: $H = F + p\cdot G = y - u^2 + p\cdot(y+u)$, $H = H(t,u,y;p)$

Merk: $p = p(t)$

Pontrjagins maksimumsprinsipp:

$$\left.\begin{aligned} &(1)\quad H'_u = 0 \\ &(2)\quad p' = -H'_y \end{aligned}\right\} \text{førsteordensbetingelser}$$

(1) H'_u: $(y - u^2 + p(y+u))'_u = -2u + p = 0$

(2) H'_y: $p' = -(1+p)$

$$\boxed{\begin{aligned} -2u + p &= 0 \\ p' + p &= -1 \end{aligned}} + \begin{cases} y' = y+u \\ y(0) = 1 \\ y(1) = e - \frac{1}{2e} + 1/2 \end{cases}$$