Plan
1 Pontryagins maksimumsprinsipp
2 Oppgaver: Optimal kontrollteori

max/min $\int_a^b F(t,y,u)\,dt$ når $\begin{cases} y' = G(t,u,y) \\ y(a) = y_0 \\ y(b) = y_1 \end{cases}$

$y = y(t)$: tilstandsvariabel

$u = u(t)$: kontrollvariabel

Metode: $H = F(t,y,u) + p \cdot G(t,y,u)$ Hamilton-funksjon

$p = p(t)$ multiplikator

Resultat: Pontryagins maksimumsprinsipp

Hvis $(y(t), u(t))$ er optimal, så har vi:

1) $H'_u = 0$

2) $p' = -H'_y$

Merk: i) Løsn. som tilfredstiller 1) og 2) ovenfor kalles normalløsninger

[ii) Unntakspkt: $H = p \cdot G(t,x,u)$, $p \neq 0$]

Se [FMEA] (Sydsæther)

Hvis H er konveks i (y,u), så er normalløsn $y^*(t)$ min

— " — konkav — " — max.

Eks: max/min $\int_0^1 y - u^2\,dt$ når $\begin{cases} y' = y + u \\ y(0) = 1 \\ y(1) = e - \frac{1}{2e} + \frac{1}{2} \end{cases}$

$F = y - u^2 \qquad G = y + u$

Normaltilfellet: $H = F + p \cdot G$
$= y - u^2 + p\cdot(y+u)$

$H'_u = -2u + p = 0$
$p' = -H'_y = -(1+p) = -1-p$

$$\left.\right\} \boxed{\begin{array}{l} p = 2u \quad (1) \\ p' + p = -1 \quad (2) \\ + \\ y' = y+u \quad (3) \\ y(0) = 1 \\ y(1) = e - \frac{1}{2e} + \frac{1}{2} \end{array}}$$

Skal finne:
$y(t) = y$
$u(t) = u$
$p(t) = p$

Alt: (2) $p' + p = -1$

$p = p_h + p_p = \underline{C_1 e^{-t} - 1}$

p_h: $p' + p = 0$
$r + 1 = 0 \quad r = -1$
$p_h = C_1 \cdot e^{-t}$

p_p: $p' + p = -1$
$p = A$:
$0 + A = -1$
$A = -1$
$p_p = -1$

(1) $u = \frac{1}{2}p$
$= \frac{1}{2}(C_1 e^{-t} - 1)$
$u = \underline{\frac{C_1}{2} \cdot e^{-t} - \frac{1}{2}}$

(3) $y' = y + \frac{1}{2}C_1 e^{-t} - \frac{1}{2}$
$y' - y = \frac{1}{2}C_1 e^{-t} - \frac{1}{2} \quad \Big| \cdot e^{-t}$

Int. faktor: $y' + a(t)y = b(t)$

$a(t) = -1 \Rightarrow \int a(t)\,dt = \int -1\,dt$
$= -t + C$

$\Rightarrow \boxed{u = e^{-t}}$ int. faktor.

$\color{red}{(y' - y)e^{-t}}$

$(y \cdot e^{-t})' = \frac{1}{2}C_1 e^{-2t} - \frac{1}{2}e^{-t}$

$y \cdot e^{-t} = \int \frac{1}{2}C_1 e^{-2t} - \frac{1}{2}e^{-t}\,dt$

$y e^{-t} = \frac{1}{2}C_1 \frac{1}{(-2)} e^{-2t} - \frac{1}{2}\frac{1}{(-1)}e^{-t} + C_2$

$y = \left(-\frac{1}{4}C_1 e^{-2t} + \frac{1}{2}e^{-t} + C_2\right)e^{t}$

$y = \underline{-\frac{1}{4}C_1 e^{-t} + \frac{1}{2} + C_2 e^{t}}$

$y = \underline{K_1 e^{-t} + K_2 e^{t} + \frac{1}{2}}$ $\quad \boxed{\begin{array}{l} K_1 = -C_1/4 \\ K_2 = C_2 \end{array}}$

$$y = K_1 e^{-t} + K_2 e^{t} + \tfrac{1}{2}$$

$y(0)=1$: $K_1 + K_2 + \frac{1}{2} = 1 \Rightarrow K_1 + K_2 = 1/2$

$y(1) = e - \frac{1}{2e} + \frac{1}{2}$: $K_1 \cdot e^{-1} + K_2 e + \cancel{\tfrac{1}{2}} = e - \frac{1}{2e} + \cancel{1/2}$

$\underline{K_1 \cdot e^{-1}} + \underline{K_2 \cdot e} = \underline{e} - \underline{\frac{1}{2} \cdot \frac{1}{e}}$

$K_2 = 1 \quad K_1 = -1/2$

$K_1 = -C_1/4 = -1/2$

$\underline{C_1 = 2}$

$$\Rightarrow \boxed{\begin{aligned} y^* &= \underline{-\tfrac{1}{2}e^{-t} + e^{t} + 1/2} \\ u^* &= \underline{e^{-t} - 1/2} \\ p^* &= \underline{2e^{-t} - 1} \end{aligned}}$$

kandidat pkt. = normal løsn.

<u>max/min?</u> $H = y - u^2 + p \cdot (y + u)$

H konveks / <u>konkav</u> i (y, u)

$H'_y = 1 + p$

$H'_u = -2u + p$

$H''_{yy} = 0 \quad H''_{yu} = 0$

$H''_{yu} = 0 \quad H''_{uu} = -2$

$H(H) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$ det $= 0$, tr $= -2 < 0$

H <u>konkav</u> i (y, u)

$\Downarrow$

$y^* = \underline{\underline{-\frac{1}{2} e^{-t} + e^{t} + 1/2}}$ gir <u>maks</u>

Alt 2:

$p = 2u$ ① $p = 2(y'-y) = 2y' - 2y$

$p' + p = -1$ $\Rightarrow p' = 2y'' - 2y'$

② $p' + p = -1$ $y' = y + u \Rightarrow u = y' - y$ ③

$(2y'' - \cancel{2y'}) + (\cancel{2y'} - 2y) = -1$

$2y'' - 2y = -1 \quad |:2$

$y'' - y = -\frac{1}{2} \Rightarrow y = y_h + y_p = \underbrace{C_1 \cdot e^{-t} + C_2 e^{t}}_{y_h} \underbrace{+ 1/2}_{y_p}$

$r^2 - 1 = 0$

Alt: Integrerende faktor → Superposisjon

$y' - y = \frac{1}{2} C_1 e^{-t} - \frac{1}{2}$

$y = y_h + y_p = \underbrace{C_2 \cdot e^{t}}_{y_h} \underbrace{- \frac{1}{4} C_1 e^{-t} + 1/2}_{y_p}$

y_h: $y' - y = 0$

$r - 1 = 0 \quad r = 1 \quad y_h = C_2 e^{t}$

y_p: $y' - y = \boxed{\frac{1}{2} C_1 e^{-t} - 1/2}$ $\quad y_p = -\frac{1}{4} C_1 e^{-t} + 1/2$

$y = A e^{-t} + B$
$y' = -A e^{-t}$

$(-A e^{-t}) - (A e^{-t} + B) = \frac{1}{2} C_1 e^{-t} - \frac{1}{2}$

$-2A e^{-t} - B = \frac{1}{2} C_1 e^{-t} - \frac{1}{2}$

$-2A = \frac{1}{2} C_1$
$-B = -1/2$

$\boxed{A = -\frac{1}{4} C_1 \quad B = 1/2}$

Euler-likning = Normalløsn av betingelsene i Pontryagins maksimumsprinsipp.

$$\max/\min \int_a^b F(t,y,y')\,dt \quad \text{når} \quad \begin{cases} y(a)=y_0 \\ y(b)=y_1 \end{cases}$$

$$= \max/\min \int_a^b F(t,y,u)\,dt \quad \text{når} \quad \begin{cases} y'=u \\ y(a)=y_0 \\ y(b)=y_1 \end{cases}$$

(1) Hamilton:

$H = F(t,y,u) + p\cdot u$

(i) $H'_u = 0 \Rightarrow F'_u + p = 0 \Rightarrow p = -F'_u = -F'_{y'}$

(ii) $p' = -H'_y$ $\qquad \Rightarrow p' = \frac{d}{dt}(-F'_{y'})$ (1) + (3)

(iii) $y' = u$

(2) $p' = -H'_y = -F'_y$

$$\frac{d}{dt}(-F'_{y'}) = -F'_y$$

$$\boxed{F'_y - \frac{d}{dt}(F'_{y'}) = 0}$$ Euler-likn.