Plan

1 Oppsumering: Lineær algebra og vektorregning

Grunnleggende

- løse lineære system vha Gauss-eliminasjon
- regne ut determinant og invers matrise
- regne med vektorer (lineær kombinasjoner, indreprodukt, lengde)
- regne med matriser (lineær kombinasjoner, matrisemultiplikasjon, transponering)

Vektorrom / lineære underrom

$\underbrace{V = \text{span}(\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r})}_{\text{lineært underrom (vektorrom)}}$: alle lin. komb. $c_1\underline{v_1} + c_2\underline{v_2} + \cdots + c_r\underline{v_r}$ av vektorene

- lineær uavhengighet : $\{\underline{v_1}, \ldots, \underline{v_r}\}$ lin. uavh

$\Updownarrow$

$\left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \underline{v_3} \,\middle|\, \cdots \,\middle|\, \underline{v_r}\right) = A$ har pivot i hver kolonne $\iff$ $x_1\underline{v_1} + x_2\underline{v_2} + \cdots + x_r\underline{v_r} = \underline{0}$ har kun triviell løsning

– basis for V: en mengde av vektorer som utspenner V og er lineært uavhengige.

Hvis $V = \text{span}(\underline{v_1}, \dots, v_r)$ og vi finner en trappeform for $A = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_r})$, så danner vektorene med pivot en basis for V.

– dimensjonen til V: antall vektorer i en basis

Eks: $\underline{v_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $v_4 = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$

$V = \text{span}(\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4})$

Basis: $A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & -1 & 1 & 0 \\ 1 & 3 & -1 & 5 \end{pmatrix} \begin{matrix} -2 \\ -1 \end{matrix} \to \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -1 & -4 \\ 0 & 1 & -2 & 3 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & -5 & -1 & -4 \end{pmatrix} \begin{matrix} \\ 5 \end{matrix} \to \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & -11 & 11 \end{pmatrix}$

$\Rightarrow \{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ basis for V. $V = \text{span}(\underline{v_1}, \underline{v_2}, \underline{v_3})$

$\underline{v_4} = c_1\underline{v_1} + c_2\underline{v_2} + c_3\underline{v_3}$ $\dim V = \underline{\underline{3}}$

Eks: Løsninger av $A \cdot \underline{x} = \underline{0}$: $W = \{\underline{x} : A\underline{x} = \underline{0}\}$

$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & -1 & 1 & 0 \\ 1 & 3 & -1 & 5 \end{pmatrix} \to \dots \to \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & -11 & 11 \end{pmatrix}$

$\underline{x_1} + 2x_2 + x_3 + 2x_4 = 0$
$\underline{x_2} - 2x_3 + 3x_4 = 0$
$-11\underline{x_3} + 11x_4 = 0$

$\underline{x_4 \text{ fri}}$

$x_3 = \underline{x_4}$ $x_2 - 2x_4 + 3x_4 = 0$ $x_1 + \cancel{2(-x_4)} + x_4 + \cancel{2x_4} = 0$
$x_2 + x_4 = 0$ $x_1 = -x_4$
$\underline{x_2 = -x_4}$

$$\left.\begin{array}{l} x_1 = -x_4 \\ x_2 = -x_4 \\ x_3 = x_4 \\ x_4 \text{ fri} \end{array}\right\} \quad \underline{x} = \begin{pmatrix} -x_4 \\ -x_4 \\ x_4 \\ x_4 \end{pmatrix} = x_4 \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = x_4 \cdot \underline{w}, \quad \underline{w} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$W = \text{span}(\underline{w})$ Basis: $\{\underline{w}\}$

$\dim W = 1$

2 Eksamen, Opps I:

a) $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 8 & 2 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{pmatrix}$ $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ lin. uavh

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & -5 & 2 \\ -6 & 15 & 0 \\ 0 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 2 & -5 & 2 \\ -6 & 15 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \Big]\,3 \to \begin{pmatrix} 2 & -5 & 2 \\ 0 & 0 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \Big]\,-4/6$$

$$\to \begin{pmatrix} 2 & -5 & 2 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ $\{\underline{u}_3, \underline{u}_4, \underline{u}_5\}$ ikke lin uavh

$x_1\underline{u}_1 + x_2\underline{u}_2 + \dots + x_5\underline{u}_5 = \underline{0}$

$$\begin{pmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 8 & 2 & -5 & 2 \\ 0 & 0 & -6 & 15 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

x_4 fri: $4x_5 = 0$ $\quad x_5 = 0$

$-6x_3 + 15x_4 = 0$ $\quad x_3 = \frac{15}{6}x_4 = \frac{5}{2}x_4$

$8x_2 + \underbrace{2x_3}_{=5x_4} - 5x_4 + \underbrace{2x_5}_{=0} = 0$

$8x_2 = -5x_4 + 5x_4 - \cancel{0}$ $\quad x_2 = 0$

$x_1 + 3x_2 = 0$ $\quad x_1 = 0$

Løsn: $\underline{x} = \begin{pmatrix} 0 \\ 0 \\ 5x_4/2 \\ x_4 \\ 0 \end{pmatrix} = x_4/2 \cdot \begin{pmatrix} 0 \\ 0 \\ 5 \\ 2 \\ 0 \end{pmatrix}$

b) Basis for $\text{Span}(\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4)$: $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 8 & 2 & -5 \\ 0 & 0 & -6 & 15 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Dimensjonen til $\text{Span}(\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4, \underline{u}_5) = \underline{\underline{4}}$

siden $\{\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_5\}$ basis

Anta om: $\text{Span}(u_1, u_2, u_3) = \text{Span}(u_2, u_4, u_5)$?

~~Vet: $\text{Span}(\underline{u}_1, \underline{u}_2, \underline{u}_3) = \text{Span}(\underline{u}_1, \underline{u}_2, \underline{u}_3, u_4)$~~

~~Hvis $\underline{u}_5$ er med i $\underbrace{\text{span}(u_1, \underline{u}_2, \underline{u}_3)}_{\dim = 3} = \text{Span}(u_1, \underline{u}_2, \underline{u}_3, \underline{u}_4) = \underbrace{\text{span}(\underline{u}_1, u_2, \underline{u}_3, \underline{u}_4, \underline{u}_5)}_{\dim 4}$~~

~~Dette er umulig; så de to lineære underrommene er ikke like.~~

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -7 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad \dim V_1 = 3$$

$V_1 = \text{span}(\underline{u}_1, \underline{u}_3, \underline{u}_5)$

$$\dim V_2 = 3 \quad \begin{pmatrix} 3 & 0 & 0 \\ 8 & -5 & 2 \\ 0 & 15 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$V_2 = \text{span}(\underline{u}_2, \underline{u}_4, \underline{u}_5)$

$V_3 = \text{Span}(\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_5)$
$\dim V_3 = 4$

$\underline{u}_2$ ikke med i $V_1 \Rightarrow \underline{\underline{V_1 \neq V_2}}$

Oppsummering fortsatt:

Egenverdier/egenvektorer

$\underset{n\times n}{A}$ $\quad A\underline{x} = \lambda \underline{x}$ $(*)$ $\quad \lambda$ egenverdi hvis $(*)$ har ikketrivielle løsn.

$\Updownarrow$ $\quad$ Metode: $\det(A - \lambda I) = 0$

$(A - \lambda I)\underline{x} = \underline{0}$

$\underline{x}$ egenvektor: $A\underline{x} = \lambda \underline{x}$

$E_\lambda = \{\underline{x} : A\underline{x} = \lambda \underline{x}\}$ egenrom

A diagonaliserbar: $P^{-1}AP = D$ diagonal for en invertibel matrise P

$\Updownarrow$

det fins en basis for $\mathbb{R}^n$ som består av egenvektorer for A

(i.e n lin. uavhengige egenvektorer)

$$P = \left(\underline{v_1} | \underline{v_2} | \cdots | \underline{v_n} \right) \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$$

A symmetrisk: A diagonaliserbar

Vi har $\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n = \det(A)$

$\lambda_1 + \lambda_2 + \ldots + \lambda_n = \operatorname{tr}(A)$

A symmetrisk $\Leftrightarrow$ $P^TAP = D$ der P er en ortogonal matrise

(A ortogonal diag.)

P ortogonal: $P^{-1} = P^T$

Kvadratiske funksjoner:

$f(x_1, x_2, \ldots, x_n)$ kvadratisk funksjon (polynom av grad 2) $\Longrightarrow$ $f(\underline{x}) = \underbrace{\underline{x}^T A \underline{x}}_{\text{grad 2}} + \underbrace{B\underline{x}}_{\text{grad 1}} + \underbrace{C}_{\text{grad 0}}$

Stasjonære pkt:

$f'(\underline{x}) = 2A\underline{x} + B^T = \underline{0}$

Kvadratisk form:

$q(\underline{x}) = \underline{x}^T A \underline{x}$

(A symmetrisk)

A/q indefinit ellers

$\Updownarrow$

stasjonære pkt er sadelpkt

A/q pos. semidefin.

$\Updownarrow$

$q(\underline{x}) \geq 0$ for alle $\underline{x}$

$\Updownarrow$

$\lambda_1, \ldots, \lambda_n \geq 0$ (egenverdiene til A)

$\Updownarrow$

stasjonære pkt er globale min

neg. semidefin.

$\Updownarrow$

$q(\underline{x}) \leq 0$

$\Updownarrow$

$\lambda_1, \ldots, \lambda_n \leq 0$

$\Updownarrow$

stasj pkt er globale max

Eksamen, Oppg 2:

$$f(x_1,x_2,x_3) = -5x_1^2 - 5x_2^2 - 5x_3^2 + 6x_1x_2 + 8x_2x_3 + 8x_1 - 6x_3 + 2017$$

a) $f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C \qquad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$A = \begin{pmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{pmatrix} \qquad B = \begin{pmatrix} 8 & 0 & -6 \end{pmatrix} \qquad C = (2017)$$

b) Kar. Likn: $|A - \lambda I| = 0$

$$\begin{vmatrix} -5-\lambda & 3 & 0 \\ 3 & -5-\lambda & 4 \\ 0 & 4 & -5-\lambda \end{vmatrix} = 0$$

$$\boxed{\begin{array}{l} -\lambda^3 - 15\lambda^2 - 50\lambda = 0 \\ \text{Sett inn } \lambda = 0, -5, -10. \end{array}}$$

$$(-5-\lambda)\cdot\left[(-5-\lambda)^2 - 4^2\right] - 3\left[3(-5-\lambda) - \cancel{4\cdot 0}\right] = 0$$

$$(-5-\lambda)\cdot\left[\lambda^2 + 10\lambda + 9\right] - 9(-5-\lambda) = 0$$

$$(-5-\lambda)\cdot\left[\lambda^2 + 10\lambda + \cancel{9} - \cancel{9}\right] = 0$$

$$(-5-\lambda)(\lambda^2 + 10\lambda) = 0$$

$$-(\lambda+5)\cdot\lambda\cdot(\lambda+10) = 0 \Rightarrow \lambda = -5, \lambda = 0, \lambda = -10$$

Vis at $\lambda = -5, -10, 0$ er egenverdiene til A.

c) $P^T A P = D$ med P ortogonal: $P = (\underline{v_1} | \underline{v_2} | \underline{v_3})$

$\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ lin. uavh. egenvektorer for A, og en ortonormal mengde

$\underline{\lambda = 0}$: $\begin{pmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{pmatrix} \underline{x} = \underline{0}$ $\quad \underline{v_1} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$ $\quad \|\underline{v_1}\| = \sqrt{50} = 5 \cdot \sqrt{2}$

$\underline{\lambda = -5}$: $\begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \underline{x} = \underline{0}$ $\quad \underline{v_2} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$ $\quad \|\underline{v_2}\| = 5$ $\quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -10 \end{pmatrix}$

$\underline{\lambda = -10}$: $\begin{pmatrix} 5 & 3 & 0 \\ 3 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix} \underline{x} = \underline{0}$ $\quad \underline{v_3} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$ $\quad \|\underline{v_3}\| = \sqrt{50} = 5 \cdot \sqrt{2}$ $\quad P = \begin{pmatrix} 3 & 4 & 3 \\ 5 & 0 & -5 \\ 4 & -3 & 4 \end{pmatrix}$

$\left.\begin{array}{l} \underline{v_1} \cdot \underline{v_2} = 0 \\ \underline{v_1} \cdot \underline{v_3} = 0 \\ \underline{v_2} \cdot \underline{v_3} = 0 \end{array}\right\}$ Hvis A er symmetrisk så er egenvektorer fra ulike egenrom alltid ortogonale.

Ortonormal: $\underline{v_1}' = \frac{1}{\sqrt{50}} \cdot \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$ $\quad \underline{v_2}' = \frac{1}{5} \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$ $\quad \underline{v_3}' = \frac{1}{\sqrt{50}} \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$

$$P = \begin{pmatrix} 3/\sqrt{50} & 4/5 & 3/\sqrt{50} \\ 5/\sqrt{50} & 0 & -5/\sqrt{50} \\ 4/\sqrt{50} & -3/5 & 4/\sqrt{50} \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -10 \end{pmatrix}$$

d) Stasj. pkt:

$$f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$$

$$f'(\underline{x}) = 2A\underline{x} + B^T = \underline{0}$$

$$2A\underline{x} = -B^T$$

$$A\underline{x} = -\frac{1}{2}B^T$$

$$\begin{pmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{pmatrix} \cdot \underline{x} = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -5 & 3 & 0 & -4 \\ 3 & -5 & 4 & 0 \\ 0 & 4 & -5 & 3 \end{array}\right) \cdot 5 \to \left(\begin{array}{ccc|c} -5 & 3 & 0 & -4 \\ 15 & -25 & 20 & 0 \\ 0 & 4 & -5 & 3 \end{array}\right) 3$$

$$\to \left(\begin{array}{ccc|c} -5 & 3 & 0 & -4 \\ 0 & -16 & 20 & -12 \\ 0 & 4 & -5 & 3 \end{array}\right) :4 \to \left(\begin{array}{ccc|c} -5 & 3 & 0 & -4 \\ 0 & -4 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_3 fri

$$-4x_2 + 5x_3 = -3$$

$$\frac{-4x_2}{-4} = \frac{-3-5x_3}{-4}$$

$$\underline{x_2 = \frac{3}{4} + \frac{5}{4}x_3}$$

$$-5x_1 + 3\left(\frac{3}{4} + \frac{5}{4}x_3\right) = -4 \quad | \cdot 4$$

$$-20x_1 + 9 + 15x_3 = -16$$

$$\frac{-20x_1}{-20} = \frac{-25-15x_3}{-20}$$

$$x_1 = \underline{\frac{5}{4} + \frac{3}{4}x_3}$$

Stasj. pkt: $\underline{\underline{\left(\frac{5}{4} + \frac{3}{4}t,\ \frac{3}{4} + \frac{5}{4}t,\ t\right)}}$

A neg. semidef. $\Rightarrow$ globale maks-pkt

siden $\lambda_1, \lambda_2, \lambda_3 \leq 0$

$$\underset{0}{\lambda_1}, \underset{-5}{\lambda_2}, \underset{-10}{\lambda_3}$$

e) $t=0$ gir $\underline{x}^* = (5/4, 3/4, 0) = \frac{1}{4}\cdot(5,3,0)$ som et av de stasjonære punktene. Siden alle stasj. pkt er maks. pkt, har alle samme verdi:

$$f_{max} = f\left(\tfrac{1}{4}(5,3,0)\right) = \left(\tfrac{1}{4}\right)^2 \cdot (5\ 3\ 0)\begin{pmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$$

$$+ \left(\tfrac{1}{4}\right)\cdot (8\ 0\ -6)\begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} + (2017)$$

$$= \frac{1}{16}\cdot(-125 - 45 + 90) + \frac{1}{4}(40) + 2017$$

$$= \frac{-80}{16} + \frac{40}{4} + 2017 = -5 + 10 + 2017 = \underline{2022}$$

<u>Antall løsn. for $f(\underline{x}) = k$:</u> Nivåkurven i høyde k

$k > 2022 = f_{max}$: $f(\underline{x}) = k$ har <u>ingen punkter</u>

$k = 2022 = f_{max}$: $f(\underline{x}) = k$ er alle maksimumspunktene $=$ stasjonære pkt $= \left(\frac{5}{4} + \frac{3}{4}t, \frac{3}{4} + \frac{5}{4}t, t\right)$, dvs. <u>uendelig mange pkt</u> (en linje)

<u>$k < 2022 = f_{max}$:</u> $f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C = k$

<u>Ortogonal diag:</u> $\underline{u}^T D \underline{u} + BP\underline{u} + C = k$

$\underline{x} = P\underline{u}$ gir $0\cdot u_1^2 - 5u_2^2 - 10u_3^2 - 10u_2 + 2017 = k$

i) $\underline{x}^T A \underline{x} = (P\underline{u})^T A (P\underline{u})$
$= \underline{u}^T P^T A P \underline{u}$
$= \underline{u}^T D \underline{u} = \lambda_1 u_1^2 + \lambda_2 u_2^2 + \lambda_3 u_3^2$

$$-5(u_2+1)^2 - 10u_3^2 = k - 2017 - 5$$
$$-5(u_2+1)^2 - 10u_3^2 = k - 2022$$
$$5(u_2+1)^2 + 10u_3^2 = 2022 - k > 0$$

ii) $B\underline{x} = BP\underline{u}$ må regnes ut:

$$(-8\ 0\ 6)\cdot\begin{pmatrix} 3/\sqrt{50} & 4/5 & 3/\sqrt{50} \\ 5/\sqrt{50} & 0 & -5/\sqrt{50} \\ 4/\sqrt{50} & -3/5 & 4/\sqrt{50} \end{pmatrix}$$

$$= \frac{1}{\sqrt{50}}\cdot(0\ \ -50\sqrt{2}\ \ 0) = \underline{(0\ -10\ 0)}$$

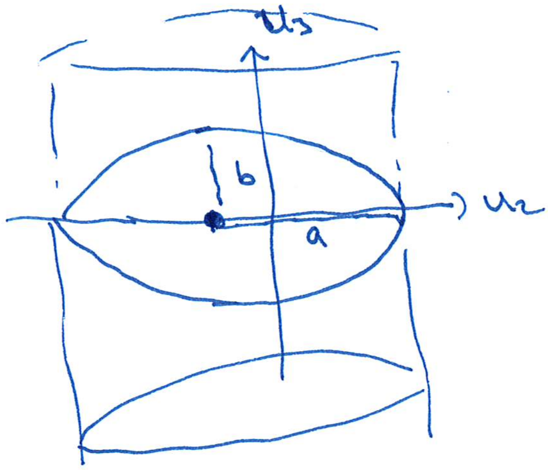
$$\boxed{\frac{5(u_2+1)^2}{2022-k} + \frac{10u_3^2}{2022-k} = 1}$$

$k < 2022$: ellipse, senter $(-1,0)$ i (u_2,u_3)-pl.

u_1: fri

<u>uendelig mange pkt</u>

K < 2022:



$$\frac{(u_2+1)^2}{(2022-k)/5} + \frac{u_3^2}{(2022-k)/10} = 1$$

Halvakser: $a = \sqrt{\frac{2022-k}{5}}$

$b = \sqrt{\frac{2022-k}{10}}$

i (u_2, u_3)-planet

u_1: fri $\Rightarrow$ $f(\underline{x}) = k$ blir en elliptisk sylinder i (u_1, u_2, u_3)-koordinatsystemet.

når $k < 2022$

uendelig mange pkt