Plan

1 Lineære underrom og span
2 Lineær uavhengighet. Dimensjon og basis.

① Lineære underrom

$\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}$ r n-vektorer

Defn. $\text{Span}(\underline{v_1}, \ldots \underline{v_r})$ = mengden av alle lineær kombinasjoner av $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}$

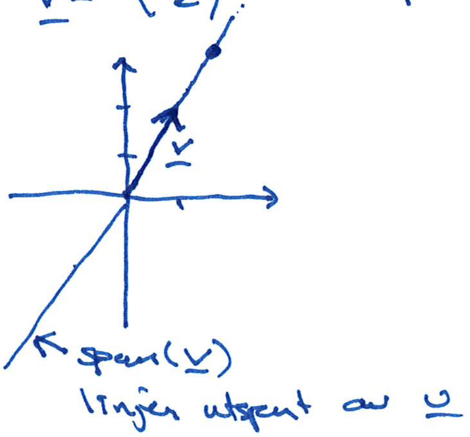
$= \{ c_1\underline{v_1} + c_2\underline{v_2} + \ldots + c_r\underline{v_r} : c_1, c_2, \ldots c_r \text{ er tall} \}$

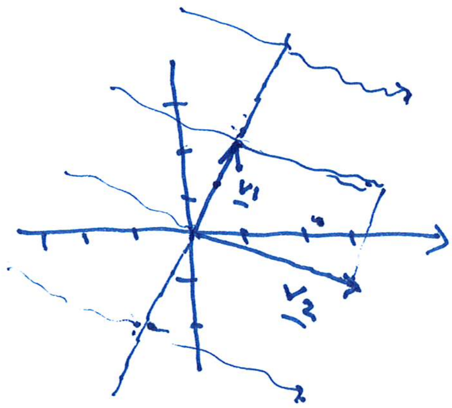
underrom av $\mathbb{R}^n$ (n-dim. koordinatsystem)

Eks:

$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$: $\text{span}(\underline{v}) = \{ c \cdot \underline{v} : c \text{ er et tall} \}$ i $\mathbb{R}^2$

$2\underline{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$





Eks: $\underline{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$: $\text{span}(\underline{v_1}, \underline{v_2}) = \{ c_1\underline{v_1} + c_2\underline{v_2} : c_1, c_2 \text{ tall} \}$

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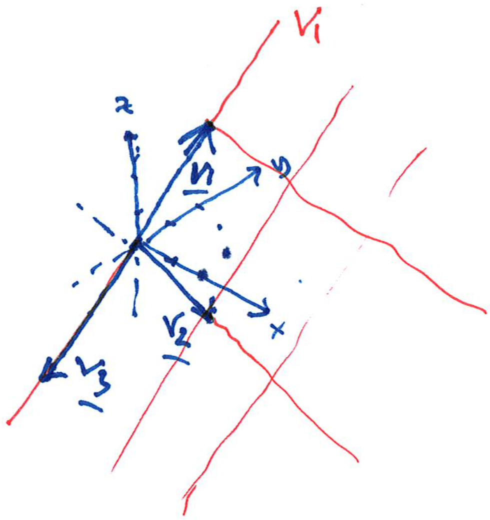
Defn. Et <u>lineær underrom</u> V av $\mathbb{R}^n$ er en mengde $V = \text{span}(\underline{v}_1, \dots, \underline{v}_r)$ utspent av r n-vektorer $\underline{v}_1, \dots, \underline{v}_r$.

Merk: Lineære underrom har følgende egenskaper:

i) $V = \text{span}(\underline{v}_1, \dots, \underline{v}_r)$ inneholder $\underline{0}$.

ii) $V = \text{span}(\underline{v}_1, \dots, \underline{v}_r)$ har ingen krumning (rettlinjet)



Eks. i) $V_1 = \text{span}(\underline{v}_1)$ $\quad \underline{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

ii) $V_2 = \text{span}(\underline{v}_1, \underline{v}_2)$ $\quad \underline{v}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

iii) $V_3 = \text{span}(\underline{v}_1, \underline{v}_3)$ $\quad \underline{v}_3 = \begin{pmatrix} -2 \\ -4 \\ -8 \end{pmatrix} = -2 \cdot \underline{v}_1$

$$V_3 = \text{span}(\underline{v}_1, \underline{v}_3) = \text{span}(\underline{v}_1) \quad : \quad c_1 \cdot \underline{v}_1 + c_3 \cdot \underline{v}_3 = c_1 \cdot \underline{v}_1 + c_3 \cdot (-2\underline{v}_1) = (c_1 - 2c_3) \cdot \underline{v}_1$$

(2) **Defn:** Vektorene $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ kalles lineært avhengige hvis (minst) én av vektorene er en lineær-kombinasjon av de andre vektorene, ellers kalles vektorene lineært uavhengige

Eks: $\underline{v}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Lin. avhengige:

$\underline{v}_1 = c \cdot \underline{v}_2$ eller $\underline{v}_2 = c \cdot \underline{v}_1$

$\Updownarrow$ $\Updownarrow$

$1 \cdot \underline{v}_1 - c \cdot \underline{v}_2 = \underline{0}$ $\quad -c \cdot \underline{v}_1 + 1 \cdot \underline{v}_2 = \underline{0}$

$(1, -c)$ løsn. $\quad (-c, 1)$ løsn.

$\boxed{x \cdot \underline{v}_1 + y \cdot \underline{v}_2 = \underline{0}}$ har ikke-trivielle løsninger $(x, y) \neq (0,0)$.

$$x \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + y \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x + 2y = 0$$
$$4x + 3y = 0$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 4 & 3 & 0 \end{array}\right) \xrightarrow{-4} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & 0 \end{array}\right)$$

$x + 2y = 0$ $\quad x = 0$

$-5y = 0$ $\quad y = 0$

Kun triviell løsn.

$\Downarrow$

$\underline{v}_1, \underline{v}_2$ lineært uavhengige

Resultat: $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ n-vektorer

i) Vektorlikningen $x_1 \cdot \underline{v}_1 + x_2 \cdot \underline{v}_2 + \ldots + x_r \cdot \underline{v}_r = \underline{0}$ har

enten i) minst én fri variabel og uendelig mange løsn. eller ii) ingen frie variabler og kun løsning $\underline{x} = (0, 0, \ldots, 0)$ (triviell løsn.)

ii) Vektorene $\underline{v}_1, \ldots, \underline{v}_r$ er lineært avhengige i tilfelle i) og lineært uavhengige i tilfelle ii).

Eks: $\underline{v_1} = \begin{pmatrix}1\\2\\1\end{pmatrix}$ $\underline{v_2} = \begin{pmatrix}2\\-1\\7\end{pmatrix}$ $\underline{v_3} = \begin{pmatrix}3\\4\\5\end{pmatrix}$

$$x \cdot \begin{pmatrix}1\\2\\1\end{pmatrix} + y\begin{pmatrix}2\\-1\\7\end{pmatrix} + z \cdot \begin{pmatrix}3\\4\\5\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \leftarrow x\underline{v_1} + y\underline{v_2} + z\underline{v_3} = \underline{0}$$

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x - y + 4z &= 0 \\ x + 7y + 5z &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & -1 & 4 & 0 \\ 1 & 7 & 5 & 0 \end{array}\right) \begin{matrix} -2 \\ \\ -1 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -2 & 0 \\ 0 & 5 & 2 & 0 \end{array}\right) \begin{matrix} \\ \\ 1 \end{matrix}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

z fri (minst én fri) $\Rightarrow$ $\underline{v_1}, \underline{v_3}, \underline{v_3}$ er lineært avhengige

$$\begin{aligned} \underline{x} + 2y + 3z &= 0 \qquad x = -2(-2z/5) - 3z = \frac{4}{5}z - \frac{15}{5}z = \underline{-11z/5} \\ -5\underline{y} - 2z &= 0 \qquad -5y = 2z \qquad \underline{y = -2z/5} \end{aligned}$$

Løsn: $(x, y, z) = \left(-\frac{11}{5}z, -\frac{2}{5}z, z\right)$

$= \left(-\frac{11}{5}t, -\frac{2}{5}t, t\right)$ t fri / parameter

Eks. $t = 5$ gir $(-11, -2, 5)$

$-11\underline{v_1} - 2\underline{v_2} + 5\underline{v_3} = \underline{0}$

$5\underline{v_3} = 11\underline{v_1} + 2\underline{v_2}$

$$\boxed{\underline{v_3} = \frac{11}{5}\underline{v_1} + \frac{2}{5}\underline{v_2}}$$

Tolkning:

$\text{span}(\underline{v_1}, \underline{v_2}, \underline{v_3}) = \text{span}(\underline{v_1}, \underline{v_2})$

Defn: La V være et lineært underrom utspent av $\underline{v}_1, \ldots, \underline{v}_r$. Hvis $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ er lineært uavhengige, kalles de en basis for V. I så fall er $\dim V = r$.

Eks: $V = \text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ $\quad \underline{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} \quad \underline{v}_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

i) $\underline{v}_1, \underline{v}_2, \underline{v}_3$ er ikke en basis for V fordi de er lineært avhengige

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 1 & 7 & 5 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

ii) $V = \text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \text{span}(\underline{v}_1, \underline{v}_2)$ siden $\underline{v}_3 = \frac{11}{5}\underline{v}_1 + \frac{2}{5}\underline{v}_2$

iii) $\underline{v}_1, \underline{v}_2$ er en basis for V fordi de er lineært uavhengige

iv) $\dim V = 2$ $\longrightarrow$

Geometrisk tolkning

$\dim V = 1$: linje
$\dim V = 2$: plan
$\dim V = 3$: hyperplan av dim 3

Resultat: $\left.\begin{matrix} \underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r \\ n\text{-vektorer} \end{matrix}\right\}$ $A = \left(\underline{v}_1 \,|\, \underline{v}_2 \,|\, \cdots \,|\, \underline{v}_r\right)$

i) $V = \text{span}(\underline{v}_1, \ldots, \underline{v}_r)$ har $\dim V = \text{rk}\, A$

ii) Vektorene som svarer til pivotkolonnene danner en basis for V.

iii) De andre vektorene er lineærkombinasjoner av basis-vektorene.

Eks:

$$\begin{aligned} 2x-6y+4z+6w &= 0 \\ 3x-11y+7z+2w &= 0 \\ x-2y+z+11w &= 0 \end{aligned}$$

Homogent lineært system

$$\begin{pmatrix} 2 & -6 & 4 & 6 \\ 3 & -11 & 7 & 2 \\ 1 & -2 & 1 & 11 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} -1 & +5 & -3 & 4 \\ 3 & -11 & 7 & 2 \\ 1 & -2 & 1 & 11 \end{pmatrix} \xrightarrow{3,\ +1}$$

$$\rightarrow \begin{pmatrix} -1 & 5 & -3 & 4 \\ 0 & 4 & -2 & 14 \\ 0 & 3 & -2 & 15 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} -1 & 5 & -3 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 3 & -2 & 15 \end{pmatrix} \xrightarrow{-3}$$

$$\rightarrow \begin{pmatrix} -1 & 5 & -3 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & 18 \end{pmatrix}$$

w fri

$\Rightarrow x = \underline{-18t}$

$-x+5t-3(9t)+4t=0$

$y-t=0 \Rightarrow \underline{y=t}$

$-2z+18t=0 \Rightarrow \underline{z=9t}$

$\underline{w=t}$

Løsn:
$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -18t \\ t \\ 9t \\ t \end{pmatrix} = t\cdot\begin{pmatrix} -18 \\ 1 \\ 9 \\ 1 \end{pmatrix}$$

$$\text{Løsn} = \text{span}(\underline{w}), \quad \underline{w} = \begin{pmatrix} -18 \\ 1 \\ 9 \\ 1 \end{pmatrix}$$

= lineært underrom av dim 1.