Plan

1 Matriser og matriseregning.
2 Lineære system på matriseform.

Repetisjon: Lineære underrom

$V = \text{span}(\underline{v_1}, \underline{v_2}, \dots, \underline{v_r})$: $\dim V = \text{rk}\,(\underline{v_1} | \underline{v_2} | \dots | \underline{v_r})$

Homogent lineært system
V = mengden av løsninger er et lineært underrom:

n = antall variabler
A = Koeffisientmatrisen

$\dim V = \#$ frie variabler

$V = \{ t_1 \cdot \underline{w_1} + t_2 \underline{w_2} + \dots + t_s \cdot \underline{w_s} : s = \#\text{frie variabler} \}$

$= \text{span}(\underline{w_1}, \underline{w_2}, \dots, \underline{w_s})$

$\dim V = n - \text{rk} A$

Eks: $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}$ $V = \text{span}(\underline{v_1}, \underline{v_2}, \underline{v_3})$ lineært underrom av $\mathbb{R}^3$

i) Er $\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ i V?

$x \cdot \underline{v_1} + y \underline{v_2} + z \underline{v_3} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$

Ja!

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 1 \\ 1 & 3 & 9 & 4 \end{array}\right) \begin{matrix} -1 \\ -1 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 2 & 8 & 2 \end{array}\right) -2$$

$\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 7 \underline{v_1} - 7 \underline{v_2} + 2 \underline{v_3}$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 2 & 4 \end{array}\right)$$

én løsning
$z = 2 \quad y = -7 \quad x = 7$

ii) **Bestem V:**

Er $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ i V?

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 2 & 4 & b \\ 1 & 3 & 9 & c \end{array}\right) \rightarrow \dots \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & * \\ 0 & 1 & 3 & * \\ 0 & 0 & 2 & * \end{array}\right)$$

én løsn. for alle a, b, c

Konklusjon: $V = \mathbb{R}^3$

Resultat:

Hvis $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ er n-vektor (i $\mathbb{R}^n$), så har vi:

$\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ er lineært uavhengige $\iff V = \text{span}(\underline{v}_1, \ldots, \underline{v}_n) = \mathbb{R}^n$

① Matriser og matriseregning

Defn. En $m \times n$-matrise A en rektangulær tabell med tall, med m rader og n kolonner.

Eks:

$$A = \begin{pmatrix} 2 & 3 & 7 \\ -1 & 2 & 0 \end{pmatrix}$$

2×3-matrise

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} \quad b_{23} = 3$$

rad 2
kolonne 3

3×3 matrise
Kvadratisk

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$d_{ij} = 0$ når $i \neq j$

3×3 matrise
diagonal

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Regneoperasjoner:

i) Addisjon / subtraksjon: $A \pm B$

(A og B har samme størrelse)

$$\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 6 \end{pmatrix}$$

ii) Skalarmultiplikasjon: $c \cdot A$

(c tall = skalar, A matrise)

$$2 \cdot \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 4 & -2 \end{pmatrix}$$

iii) Matrisemultiplikasjon: $\underset{m\times n}{A}\cdot\underset{n\times p}{B} \rightsquigarrow \underset{m\times p}{AB}$

(# kolonner i A = # rader i B)

Eks:
$$\underset{2\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}} \cdot \underset{2\times 2}{\begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}} = \begin{pmatrix} 2 & 9 \\ 6 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \underline{v_1} \\ \hline \underline{v_2} \end{pmatrix} \cdot \left(\underline{w_1} \middle| \underline{w_2} \right) = \begin{pmatrix} \underline{v_1}\cdot\underline{w_1} & \underline{v_1}\cdot\underline{w_2} \\ \underline{v_2}\cdot\underline{w_1} & \underline{v_2}\cdot\underline{w_2} \end{pmatrix}$$

Eks:
$$A = \underset{2\times 3}{\begin{array}{ccc} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & 4 \end{pmatrix} \\ \underline{v_1} \quad \underline{v_2} \quad \underline{v_3} \end{array}} \cdot B = \underset{3\times 2}{\begin{pmatrix} 7 & 1 \\ 2 & 0 \\ -1 & 0 \end{pmatrix}} = \overset{A\cdot B}{\begin{pmatrix} 11 & 1 \\ -6 & 0 \end{pmatrix}} \neq BA$$

$$= \left(* \middle| * \right)$$

$7\underline{v_1} + 2\underline{v_2} - 1\cdot\underline{v_3}$, $1\cdot\underline{v_1} + 0\,\underline{v_2} + 0\cdot\underline{v_3}$

Merk:

i) $A\cdot(B+C) = AB + AC$
$(A+B)\cdot C = AC + BC$
$A(BC) = (AB)C$
men
$$\boxed{AB \neq BA}$$

ii) Hvis A er kvadratisk $(n\times n)$ så har vi:
$A^2 = A\cdot A \quad A^3 = A^2\cdot A \quad \ldots$

Eks: $(A+B)^2 = (A+B)\cdot(A+B)$
$= A\cdot A + B\cdot A + AB + B\cdot B$
$= A^2 + BA + AB + B^2$

iii) Hvis $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ og $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ så har vi

$\underset{2\times 1}{A} \cdot \underset{2\times 1}{A}$ ikke def. $\underline{v} \cdot \underline{v} = 1\cdot 1 + 2\cdot 2 = 5$

Men: $A^T \cdot A = (1 \;\; 2) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1\cdot 1 + 2 \cdot 2 = 5$

iv) Transponering

$\underset{m\times n \text{ matrise}}{A} \rightsquigarrow \underset{n \times m}{A^T}$

rader i $A \to$ kolonner i A^T
kolonne i $A \to$ rader i A^T

Eks: $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $A^T = (1 \;\; 2)$

$A = \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}$ $A^T = \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 3 \\ 7 & -1 \end{pmatrix}$ $A^T = \begin{pmatrix} 1 & 7 \\ 3 & -1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \end{pmatrix}$ $A^T = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & -1 \end{pmatrix}$

Defn: A er symmetrisk hvis $A^T = A$.

Merk:
$$(A \pm B)^T = A^T \pm B^T$$
$$(c \cdot A)^T = c \cdot A^T$$
$$(A \cdot B)^T = B^T \cdot A^T$$
$$(A^T)^T = A$$

Identitetsmatrisen:

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Merk: $A \cdot I = A$

$I \cdot A = A$

Eks: $\begin{pmatrix} 2 & 3 \\ 7 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(1 \cdot \underline{v}_1 + 0 \cdot \underline{v}_2 \;\middle|\; 0 \cdot \underline{v}_1 + 1 \cdot \underline{v}_2 \right)$

$\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \underline{v}_1$, $\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \underline{v}_2$

$= \left(\underline{v}_1 \middle| \underline{v}_2 \right) = \begin{pmatrix} 2 & 3 \\ 7 & -1 \end{pmatrix}$

(2) Lineære system på matriseform

$m \times n$ lin. system: variabler $x_1, x_2, \ldots, x_n$

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12} \cdot x_2 + \cdots + a_{1n} \cdot x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn} \cdot x_n &= b_m \end{aligned}} \iff \boxed{A \cdot \underline{x} = \underline{b}}$$

Matriseformen til det lin. sys.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Koeffisientmatrisen $(m \times n)$ — n-vektor — m-vektor

Eks:

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} x+y+z \\ x+2y+4z \\ x+3y+9z \end{pmatrix}}_{A\underline{x}} = \underbrace{\begin{pmatrix} 3 \\ 7 \\ 13 \end{pmatrix}}_{\underline{b}}$$