Plan

1 Egenverdier og egenvektorer
2 Diagonalisering av matriser

Repetisjon:

$\{\underline{v}_1, \underline{v}_2, \ldots, v_n\}$ m-mengde av vektorer

$A = (\underline{v}_1 | \underline{v}_2 | \ldots | \underline{v}_n)$ $m \times n$-matrise

Defn:
Mengden kalles ortogonal hvis $\underline{v}_i \cdot \underline{v}_j = 0$ når $i \neq j$, og ortonormal hvis $\underline{v}_i \cdot \underline{v}_j = 0$ for $i \neq j$ og $\underline{v}_i \cdot \underline{v}_i = 1 \Leftrightarrow \|v_i\| = 1$

Defn.:
A ortogonal hvis $A^{-1} = A^T$

$$\boxed{\underbrace{\{\underline{v}_1, \ldots, \underline{v}_n\}}_{\text{ortonormal mengde}} \Leftrightarrow \underbrace{A = (\underline{v}_1 | \ldots | \underline{v}_n)}_{\text{ortogonal}}}$$

$\underline{v} \neq \underline{0} \rightsquigarrow \frac{1}{\|v\|} \cdot \underline{v}$ normalisering

Lengden = 1

3.21 [DA] Vis at hvis $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ er en ortonormal mengde så er vektorene lineært uavhengige.

Bevis:
Anta $\{\underline{v}_1 \ldots \underline{v}_n\}$ ortonormal mengde dvs: $\boxed{\begin{array}{l} \underline{v}_i \cdot \underline{v}_i = 1 \\ \underline{v}_i \cdot \underline{v}_j = 0 \quad i \neq j \end{array}}$

se på vektorlikningen

$x_1 \cdot \underline{v}_1 + x_2 \cdot \underline{v}_2 + \ldots + x_n \cdot \underline{v}_n = \underline{0} \quad | \cdot \underline{v}_i$

$x_1 (\underline{v}_1 \cdot \underline{v}_i) + x_2 \cdot (\underline{v}_2 \cdot \underline{v}_i) + \ldots + x_n (\underline{v}_n \cdot \underline{v}_i) = \underline{0} \cdot \underline{v}_i$

$x_i \cdot 1 = 0 \quad x_i = 0$

$\Rightarrow x_1 = x_2 = \ldots = x_n = 0 \Rightarrow$ Vektorene er lineært uavhengige siden vi kun har den trivielle løsn. $\underline{x} = \underline{0}$ $\square$

Alt. basis: $\{\underline{u}_1, \dots, \underline{u}_n\}$ ortonormal mengde $\Rightarrow$ $A = (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_n)$ ortogonal matrise, dvs $A^{-1} = A^T$

$\Rightarrow$ A er invertibel, $|A| \neq 0$ $\Rightarrow$ $A = \left(\underline{v}_1 \middle| \underline{v}_2 \middle| \dots \middle| \underline{v}_n \right)$ har pivot i alle kolonnene (dvs $\text{rk}\, A = n$)

$\Rightarrow$ $A \cdot \underline{x} = \underline{0}$ $\Leftrightarrow$ $x_1 \underline{v}_1 + \dots + x_n \underline{v}_n = \underline{0}$ har én løsn. $\underline{x} = \underline{0}$

Merk: $V = \text{span}(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n)$ har basis $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ siden $A = (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_n)$ har pivot i hver kolonne.

$\Downarrow$

$V = \mathbb{R}^n$

① Egenverdier og egenvektorer

(lambda, gresk l – et tall)

A
$n\times n$-matrise

Defn: i) λ er en egenverdi for A hvis

$$\boxed{A\underline{x} = \lambda x}$$

har løsninger $\underline{x} \neq \underline{0}$ (ikke-trivielle løsn.)

ii) $\underline{x}$ er en egenvektor for A (med egenverdi λ) hvis

$$\boxed{A\underline{x} = \lambda \underline{x}} \quad \text{for et tall } \lambda$$

og $\underline{x} \neq \underline{0}$.

Eks:

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}; \quad \underline{x} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}: \quad A\cdot\underline{x} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \underbrace{\begin{pmatrix} 27 \\ 31 \end{pmatrix} = \lambda\cdot\begin{pmatrix} 3 \\ 7 \end{pmatrix}}_{\text{ingen løsn.}}$$

ikke egenvektor for A

$$\underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}: \quad A\underline{x} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \lambda\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\lambda = 5}$$

egenvektor med egenverdi $\lambda = 5$

Metode for å finne egenverdier:

$A\underline{x} = \lambda\underline{x}$ har ikke-trivielle løsn.

$A\underline{x} - \lambda\underline{x} = \underline{0}$

$A\underline{x} - \lambda I\underline{x} = \underline{0}$

$(A - \lambda I)\underline{x} = \underline{0}$ har ikke-trivielle løsn.

Eks: $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\cdot\begin{pmatrix} x \\ y \end{pmatrix}$$

$$2x + 3y = \lambda x$$
$$x + 4y = \lambda y$$

$$2x - \lambda x + 3y = 0$$
$$x + 4y - \lambda y = 0$$

$$\begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} \longrightarrow \begin{array}{l} (2-\lambda)x + 3y = 0 \\ x + (4-\lambda)y = 0 \end{array}$$

$A\underline{x} = \lambda \underline{x}$ har ikke-trivelle løsn

$\Updownarrow$

$(A - \lambda I)\underline{x} = 0$ — " —

$\Updownarrow$

$\boxed{|A - \lambda I| = 0 \quad \text{Karakteristisk likning}}$

Egenverdiene til A = løsningene av likningen.

Eks:

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(4-\lambda) - 3 \cdot 1 = 0$$

$$\underline{\lambda^2 - 6\lambda + 5 = 0}$$

$$\underline{\underline{\lambda = 1}}, \quad \underline{\underline{\lambda = 5}}$$

Eks:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

Egenverdier:

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \leftarrow \left\{ \begin{array}{l} \text{Kar. likn.} \\ |A - \lambda I| = 0 \end{array} \right.$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$ad - \underline{a\lambda} - \underline{d\lambda} + \underline{\lambda^2} - bc = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{\text{tr}\,A}\lambda + \underbrace{(ad-bc)}_{|A|} = 0$$

spor/trace til A = summen på diagonalen

Generelt: A $n \times n$ $\Longrightarrow$ $|A - \lambda I| = 0$ er en polynomlikning av grad n.

Metode for å finne egenvektorene

Anta at vi vet at λ er en egenverdi for A:

$A\underline{x} = \lambda \underline{x}$

$(A - \lambda I)\underline{x} = \underline{0}$ ← homogent lineært system med minst en frihetsgrad

$\wr$

Løs vha Gauss ← uendelig mange løsn. for hver λ

Eks: $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ $\quad \lambda = 1, \lambda = 5$ egenverdiene til A

$\underline{\lambda = 1}$: $\begin{pmatrix} 2-1 & 3 \\ 1 & 4-1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\left(\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad$ $x + 3y = 0$, $x = -3y$

y fri

Egenvektorer for $\lambda = 1$:

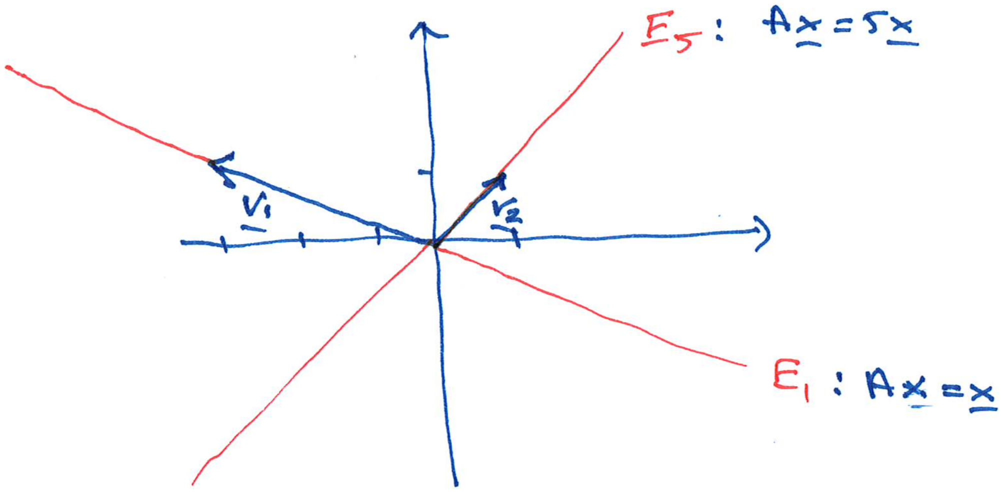
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3y \\ y \end{pmatrix} = \begin{pmatrix} -3t \\ t \end{pmatrix} = t \cdot \underline{\begin{pmatrix} -3 \\ 1 \end{pmatrix}}$ $\quad$ $\boxed{\underline{v_1} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad E_1 = \text{span}(\underline{v_1})}$

E_λ: alle egenvektorer med egenverdi λ

$\underline{\lambda = 5}$: $\begin{pmatrix} 2-5 & 3 \\ 1 & 4-5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad$ $\left(\begin{array}{cc|c} -3 & 3 & 0 \\ 1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad$ $-3x + 3y = 0$

y fri

$\underline{E_5}$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} = t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad$ $\boxed{\underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad E_5 = \text{span}(\underline{v_2})}$



$\underline{v_1}$ basis for E_1

$\underline{v_2}$ basis for E_5

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

$$\underbrace{\underline{v_1} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\text{lineært uavh.}}$$

$$\underline{v} = 2 \cdot \underline{v_1} + 3\underline{v_2} = 2 \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$$

$$A\underline{v} = A \cdot (2\underline{v_1} + 3\underline{v_2}) = A \cdot 2\underline{v_1} + A \cdot 3\underline{v_2}$$

$$= 2 \cdot \underbrace{A\underline{v_1}}_{= 1 \cdot \underline{v_1}} + 3 \cdot \underbrace{A\underline{v_2}}_{= 5\underline{v_2}} = 2\underline{v_1} + 15\underline{v_2}$$

$$= 2 \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix} + 15 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 9 \\ 17 \end{pmatrix}}}$$

$\underline{w}$ vilkårlig vektor : $\underline{w} = x \cdot \underline{v_1} + y \cdot \underline{v_2}$

$$\Updownarrow$$

$$\begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \underline{w}$$

Én løsn for alle $\underline{w}$

② Diagonalisering

A $n \times n$-matrise

Defn: A er <u>diagonaliserbar</u> hvis det finns en diagonal matrise D og en invertibel matrise P slik at

$$P^{-1} A \cdot P = D$$

$$\Updownarrow$$

$$A \cdot P = P \cdot D \quad , \quad |P| \neq 0$$

Resultat: $\boxed{\text{A er diagonaliserbar hvis og bare hvis A har n lineært uavhengige egenvektorer } \underline{v_1}, \ldots, \underline{v_n}.}$

Egenvektorer:

$A\underline{v_1} = \lambda_1 \cdot \underline{v_1}$

$A \cdot \underline{v_2} = \lambda_2 \cdot \underline{v_2}$

$\vdots$

$A\underline{v_n} = \lambda_n \cdot \underline{v_n}$

Bevis: $P = (\underline{v_1} | \underline{v_2} | \ldots | \underline{v_n})$ $n \times n$-matrise, $|P| \neq 0$

$$A \cdot P = A \cdot (\underline{v_1} | \underline{v_2} | \ldots | \underline{v_n})$$
$$= (A\underline{v_1} | A\underline{v_2} | \ldots | A\underline{v_n})$$
$$= (\lambda_1 \underline{v_1} | \lambda_2 \underline{v_2} | \ldots | \lambda_n \underline{v_n})$$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$ $n \times n$-matrise diagonal

$$PD = (\underline{v_1} | \underline{v_2} | \ldots | \underline{v_n}) \cdot \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ \vdots & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$= (\lambda_1 \underline{v_1} | \lambda_2 \underline{v_2} | \ldots | \lambda_n \underline{v_n})$$

$$\Downarrow$$

$$AP = PD. \qquad \square$$