Plan

1 Diagonalisering av matriser
2 Ortogonal diagonalisering

Repetisjon:

$\underset{n\times n}{A}$

Defn: λ egenverdi for A hvis $A\underline{v} = \lambda \underline{v} \Leftrightarrow (A-\lambda I)\underline{v} = \underline{0}$ har ikke-trivielle løsn ($\underline{v} \neq \underline{0}$)

λ egenverdi $\Leftrightarrow$ $\underset{\text{kar. lik.}}{|A-\lambda I| = 0}$

$E_\lambda = \{\underline{v} : A\underline{v} = \lambda\underline{v}\}$ for en egenverdi λ

egenrommet til A med egenverdi λ

$\underline{v}$ egenvektor for A med egenverdi λ $\Leftrightarrow$ $\underline{v} \in E_\lambda$ med $\underline{v} \neq \underline{0}$

Defn. A er diagonaliserbar hvis det fins en diagonal matrise D og en invertibel matrise P slik at

$$P^{-1}AP = D$$

Metode: Finne egenverdier og egenvektorer for A: $A\underline{v}_i = \lambda_i \underline{v}_i$

$$P = \left(\underline{v_1} \middle| \underline{v_2} \middle| \dots \middle| \underline{v_n}\right) \Rightarrow P^{-1}AP = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$$

P invertibel $\Leftrightarrow$ $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$ lin. uavh.

$$\boxed{\begin{array}{c} A \text{ diagonaliserbar} \\ \Updownarrow \\ A \text{ har } n \text{ lineært uavhengige egenvektorer} \end{array}}$$

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Eks: $A = \begin{pmatrix} 6 & 3 \\ -4 & -1 \end{pmatrix}$

Egenverdier: $|A-\lambda I| = 0$

$$\begin{vmatrix} 6-\lambda & 3 \\ -4 & -1-\lambda \end{vmatrix} = 0 \qquad \lambda^2 - tr(A)\cdot\lambda + det(A) = 0$$

$$(6-\lambda)(-1-\lambda) - 3(-4) = 0$$

$$\lambda^2 - 5\lambda + 6 = 0 \qquad \lambda_1 = 2, \ \lambda_2 = 3$$

Egenvektorer:

$\lambda = 2$: E_2

$A-\lambda I$

$$\begin{pmatrix} 4 & 3 \\ \cancel{-4} & \cancel{-3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$4x + 3y = 0$, y fri $x = -3y/4$

$$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3y/4 \\ y \end{pmatrix} = y/4 \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

Basis for E_2: $\underline{v_1} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$\lambda = 3$: E_3

$$\begin{pmatrix} 3 & 3 \\ \cancel{-4} & \cancel{-4} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$3x + 3y = 0$, y fri $x = -y$

$$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Basis for E_3: $\underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

To ~~[illegible]~~ lineært uavhengige egenvektorer: $\underline{v_1}, \underline{v_2}$

$$\begin{vmatrix} -3 & -1 \\ 4 & 1 \end{vmatrix} = -3 + 4 = 1 \neq 0$$

I praksis gjør vi følgende:

i) Finner en basis for hvert enkelt egenrom

ii) Slår sammen basisvektorene fra de ulike egenrommene

$\Rightarrow$ lineært uavhengige egenvektorer, og en maksimal slik mengde

Merk: Egenvektor fra ulike egenrom er alltid lineært uavhengige.

Fordi:

$A\underline{v_1} = \lambda_1 \underline{v_1}$ & $\lambda_1 \neq \lambda_2$

$A\underline{v_2} = \lambda_2 \cdot \underline{v_2}$

$x_1 \cdot \underline{v_1} + x_2 \cdot \underline{v_2} = \underline{0}$ | $A\cdot$

$A x_1 \underline{v_1} + A x_2 \underline{v_2} = A \cdot \underline{0}$

$x_1 \cdot (\lambda_1 \underline{v_1}) + x_2 (\lambda_2 \underline{v_2}) = \underline{0}$

$\lambda_1 \cdot$: $x_1 \cdot \lambda_1 \underline{v_1} + x_2 \lambda_1 \underline{v_2} = \underline{0}$

trekker fra: $x_2(\lambda_2 - \lambda_1)\underline{v_2} = \underline{0}$

$\Rightarrow x_2 = 0 \Rightarrow x_1 = 0$

$\Rightarrow$ Lin. uavhengige vektorer

1 eks: $P = \begin{pmatrix} -3 & -1 \\ 4 & 1 \end{pmatrix}$ og $P^{-1}AP = D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

$\uparrow \underline{v}_1 \quad \uparrow \underline{v}_2 \qquad\qquad \uparrow \lambda_1 \quad \uparrow \lambda_2$

Diagonalisering av A.

Eks:

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\lambda^2 - \cancel{0\lambda} + 1 = 0$

$\lambda^2 = -1$

ingen egenverdier

A ikke diagonaliserbar

"for få egenverdier"

Eks:

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda_1 = \lambda_2 = 1$

$\lambda = 1$: E_1

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$y = 0$, x fri

$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Basis for E_1: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$P = \begin{pmatrix} 1 & ? \\ 0 & ? \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

A ikke diagonaliserbar

"for få egenvektorer"

Resultat: A $n \times n$-matrise

i) A er diagonaliserbar
$\Leftrightarrow$ A har n lineært uavhengige egenvektorer

ii) A har n forskjellige egenverdi $\Rightarrow$ A diag.bar.

iii) A symmetrisk $\Rightarrow$ A diag. bar

② Ortogonal diagonalisering

$\underset{n\times n}{A}$ Defn: A kalles ortogonal diagonaliserbar hvis det finnes en ortogonal matrise P slik at

$$P^{-1}AP = D \text{ er diagonal}$$

$$P^T AP$$

Minner om:

$P = (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_n)$

$P^{-1} = P^T$

$\Updownarrow$

$\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ er en ortonormal mengde

$\left(\begin{array}{ll} \underline{v}_i \cdot \underline{v}_j = 0 & i \neq j \\ \underline{v}_i \cdot \underline{v}_i = 1 & \end{array} \right)$

I så fall kalles P en ortogonal matrise.

Resultat:

$$\boxed{\begin{array}{c} A \text{ ortogonal diagonaliserbar} \\ \Updownarrow \\ A \text{ symmetrisk} \end{array}}$$

Merk: A ortogonal diagonaliserbar $\Leftrightarrow$ det fins en ortonormal mengde av n egenvektor for A

Eks: $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$

Symmetrisk 3x3

Egenverdier:

$$\begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0 : \quad +(3-\lambda)\cdot \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\cdot(\lambda^2 - 6\lambda + 8) = 0$$

$$(3-\lambda)\cdot(\lambda-2)(\lambda-4) = 0$$

$$\underline{\lambda_1 = 3} \quad \underline{\lambda_2 = 2} \quad \underline{\lambda_3 = 4}$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

E_3: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad x = 0$, $z = 0$, y fri $\quad \underline{v} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\underline{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

E_2: $\left(\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 1 \end{array}\right)$ $\quad x + z = 0$, $y = 0$, z fri $\quad \underline{v} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

E_4: $\left(\begin{array}{ccc} -1 & 0 & 1 \\ 0 & -1 & 0 \\ \hline 1 & 0 & -1 \end{array}\right)$ $\quad -x + z = 0$, $-y = 0$, z fri $\quad \underline{v} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

a) $P = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$: $P^{-1}AP = D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ vanlig diagonalisering

b) Er $\underline{v_1}, \underline{v_2}, \underline{v_3}$ en ortonormal mengde?

$\underline{v_1} \cdot \underline{v_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0 + 0 + 0 = 0$

$\underline{v_1} \cdot \underline{v_3} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 + 0 + 0 = 0$

$\underline{v_2} \cdot \underline{v_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = -1 + 0 + 1 = 0$

ortogonal mengde: ✓

$\underline{v_1} \cdot \underline{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1 \qquad \|v_1\| = \sqrt{1} = 1$

$\underline{v_2} \cdot \underline{v_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 2 \qquad \|\underline{v_2}\| = \sqrt{2} \neq 1$

$\underline{v_3} \cdot \underline{v_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2 \qquad \|\underline{v_3}\| = \sqrt{2} \neq 1$

ortonormal mengde: nei

Normalisering: $\underline{w_1} = \frac{1}{\|\underline{v_1}\|} \cdot \underline{v_1} = (0, 1, 0)$

$\underline{w_2} = \frac{1}{\|\underline{v_2}\|} \cdot \underline{v_2} = \frac{1}{\sqrt{2}}(-1, 0, 1)$

$\underline{w_3} = \frac{1}{\|\underline{v_3}\|} \cdot \underline{v_3} = \frac{1}{\sqrt{2}}(1, 0, 1)$

Ortogonal diagonalisering

$P = \begin{pmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Merk:
Hvis A er symmetrisk, så er egenvektorer i forskjellige egenrom alltid ortogonale.