Plan

1. Kvadratiske former på matriseform
2. Definithet av kvadratiske former

Rep: A $n \times n$

$$A \text{ diag.bar} \Leftarrow P^{-1}AP = D \Leftarrow A \text{ symmetrisk}$$

$$A \text{ ort. diag.bar} \Leftrightarrow P^TAP = D \text{ med } P^T = P^{-1} \Leftrightarrow A \text{ symmetrisk}$$

Anvendelser: i) Kvadratiske former

ii) Beregne A^N (N stor):

$$P^{-1}AP = D \quad | \cdot P^{-1}$$

$$P^{-1}A = DP^{-1} \quad | P \cdot$$

$$A = PDP^{-1}$$

$$\Downarrow$$

$$A^N = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\ldots(\cancel{P}DP^{-1})$$

$$A^N = PD^NP^{-1}$$

Husk:

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \Rightarrow D^N = \begin{pmatrix} \lambda_1^N & & & 0 \\ & \lambda_2^N & & \\ & & \ddots & \\ 0 & & & \lambda_n^N \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 6 & 3 \\ -4 & -1 \end{pmatrix}$ $\quad P = \begin{pmatrix} -3 & -1 \\ 4 & 1 \end{pmatrix}$ $\quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

$$P^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix}$$

$$A^{27} = \begin{pmatrix} -3 & -1 \\ 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{27} \cdot \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2^{27} & 0 \\ 0 & 3^{27} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} = \ldots$$

1 Kvadratiske former

Defn: En funksjon $f(x_1, x_2, \dots, x_n)$ kalles en <u>kvadratisk form</u> hvis f er en polynomfunksjon der alle leddene i funksjonsuttrykket har grad 2.

Eks: $f(x) = ax^2 \quad \begin{cases} a > 0 & \cup \\ a < 0 & \cap \end{cases}$ $\qquad$ <u>$n=1$</u>

$f(x,y) = ax^2 + bxy + cy^2$ $\qquad$ <u>$n=2$</u>

$f(x,y,z) = ax^2 + bxy + cxz + dy^2 + eyz + fz^2$ $\qquad$ <u>$n=3$</u>

Eks: $f(x,y) = 2x^2 - 6xy + 4y^2$

$$A = \begin{array}{cc} \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} & \begin{array}{c} x \\ y \end{array} \\ \begin{array}{cc} x & \;\; y \end{array} & \end{array}$$

den symmetrisk matrisen A til den kvadratiske formen $f(x,y)$

$$(x \;\; y) \cdot \underbrace{\begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}} = (x \;\; y) \cdot \begin{pmatrix} 2x - 3y \\ -3x + 4y \end{pmatrix}$$

$$= \big(x(2x - 3y) + y(-3x + 4y) \big)$$

$$= \big(2x^2 - 3xy - 3yx + 4y^2 \big)$$

$$= \big(2x^2 - 6xy + 4y^2 \big) = \big(f(x,y) \big)$$

Merk: Korrespondanse mellom kvadratiske former i n variabler og $n \times n$ symmetriske matriser

$$A \longleftrightarrow f(x_1,\dots,x_n) = \underline{x}^T \cdot A \cdot \underline{x} \quad \text{med } \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

A: symm. $n \times n$-matrise

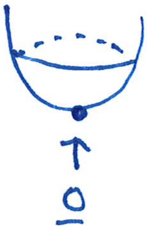
$f(x_1,\dots,x_n)$: Kvadr. form i n variabler

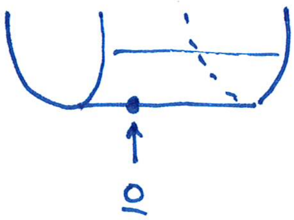
Defn: En kvadratisk form $f(x_1,x_2,\dots,x_n)$ kalles

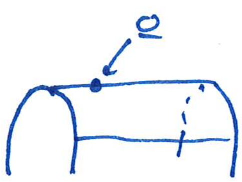
Hoved-typer:

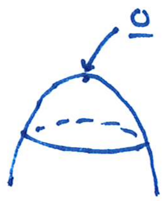
- positiv definit hvis $f(\underline{x}) > 0$ for alle $\underline{x} \neq \underline{0}$
- positiv semidefinit hvis $f(\underline{x}) = f(x_1,x_2,\dots,x_n) \geq 0$ for alle $\underline{x}$
- negativ definit hvis $f(\underline{x}) < 0$ for alle $\underline{x} \neq \underline{0}$
- negativ semidefinit hvis $f(\underline{x}) \leq 0$ for alle $\underline{x}$
- indefinit hvis f tar både pos. og neg. verdier
  $=$ f er hverken pos. eller neg. semidefn.

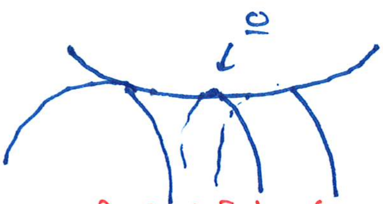
Legg merke til at $f(0,0,\dots,0) = 0$ for alle kvadratiske former.


positiv defn.
pos. semi-defn.


pos. semidefn.


neg. semidefn.


neg. defn.
neg. semidefn.


indefinit (sadelpkt)

Eks: $f(x,y) = 2x^2 - 6xy + 4y^2 \Leftrightarrow A = \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix}$

kvadr. form

symm. matrisen

$f(0,0) = 0$
$f(1,1) = 0$
$f(1,0) = 2$
$f(0,1) = 4$

pos. semidefin?

Egenverdier:

$$\begin{vmatrix} 2-\lambda & -3 \\ -3 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 6\lambda - 1 = 0$$

$$\lambda = \frac{6 \pm \sqrt{36+4}}{2} = 3 \pm \frac{\sqrt{40}}{2} = 3 \pm \sqrt{10}$$

$\lambda_1 = 3+\sqrt{10} > 0 \qquad \lambda_2 = 3-\sqrt{10} < 0 \Rightarrow f$ <u>indefinit</u>

Resultat:
Hvis $f(\underline{x})$ er en kvadratisk form i n variabler, og A er den symmetriske matrisen til f, med egenverdier $\lambda_1, \lambda_2, \ldots, \lambda_n$, så har vi:

$\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n > 0 \Leftrightarrow f$ pos. defin.
$\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0 \Leftrightarrow f$ pos. semidefin.
$\lambda_1, \lambda_2, \ldots, \lambda_n < 0 \Leftrightarrow f$ neg. defin.
$\lambda_1, \lambda_2, \ldots, \lambda_n \leq 0 \Leftrightarrow f$ neg. semidefin.
A har både pos. og neg. egenverdier $\Leftrightarrow f$ indefinit

Forklaring:

$f(\underline{x}) = \underline{x}^T A \underline{x}$

Vet at A har en ortogonal diag:

$P^T A P = D$ med $P^{-1} = P^T$, D diag.

$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$

Variabelskifte:

$$\left.\begin{aligned} &\underline{x} = P\underline{z} \\ &\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = P \cdot \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \\ &\Updownarrow \\ &P^{-1} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \\ &P^T \cdot \underline{x} = \underline{z} \end{aligned}\right\}$$

$$\begin{aligned} f(\underline{x}) &= \underline{x}^T A \underline{x} \\ &= (P\underline{z})^T A (P\underline{z}) \\ &= \underline{z}^T P^T A P \underline{z} = \underbrace{\underline{z}^T (P^T A P) \underline{z}} \\ &= f^{ny}(\underline{z}) = \underline{z}^T D \underline{z} \end{aligned}$$

$$f^{ny}(\underline{z}) = \lambda_1 z_1^2 + \lambda_2 z_2^2 + \dots + \lambda_n z_n^2$$

Merk: $f(x, y, z) = ax^2 + by^2 + cz^2$

$a, b, c > 0$: "$2x^2 + 3y^2 + 7z^2$" pos. defin.

$a, b > 0, c = 0$: "$2x^2 + 3y^2 + 0 \cdot z^2$" pos semidefin., $f(0,0,1) = 0$

$a, b, c < 0$: "$-2x^2 - 3y^2 - 7z^2$" neg. defin

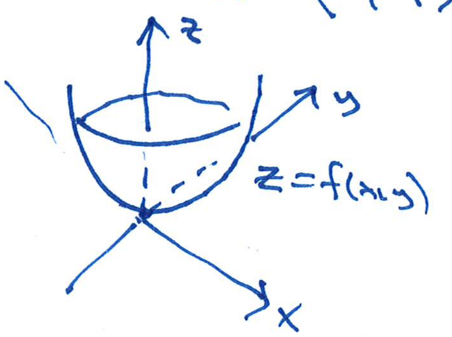
$a > 0, b, c < 0$: "$2x^2 - 3y^2 - 7z^2$" $f(1,0,0) = 2$, $f(0,1,0) = -3$ indefin.

Generelt: Vi kan alltid bruke den ortogonale diagonaliseringen til å skrive om

$$f(x_1, \dots, x_n) = f^{ny}(z_1, z_2, \dots, z_n) = \lambda_1 \cdot z_1^2 + \lambda_2 z_2^2 + \dots + \lambda_n z_n^2$$

Ex: $f(x,y) = 7x^2 - 8xy + 13y^2$

$$A = \begin{pmatrix} 7 & -4 \\ -4 & 13 \end{pmatrix}$$



$$\lambda^2 - 20\lambda + (91-16) = 0$$

$$\lambda^2 - 20\lambda + 75 = 0$$

$\lambda_1 = 5,\quad \lambda_2 = 15 \Rightarrow f$ pos. def.

E_5: $\begin{pmatrix} 2 & -4 \\ \cancel{-4} & \cancel{8} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad x - 2y = 0$, y fri

$\underline{v} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y\cdot\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\|\underline{v}_1\| = \sqrt{2^2+1^2} = \sqrt{5}$

E_{15}: $\begin{pmatrix} -8 & -4 \\ \cancel{-4} & \cancel{-2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad 2x + y = 0$, y fri

$\underline{v} = \begin{pmatrix} -y/2 \\ y \end{pmatrix} = \frac{y}{2}\cdot\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $\quad \underline{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\|\underline{v}_2\| = \sqrt{5}$

$$\underbrace{\underline{w}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \underline{w}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}}_{\text{ortonormal mengde egenvektorer}}$$

$$f^{ny}(u,v) = 5u^2 + 15v^2$$

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 15 \end{pmatrix}$$

$$P = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

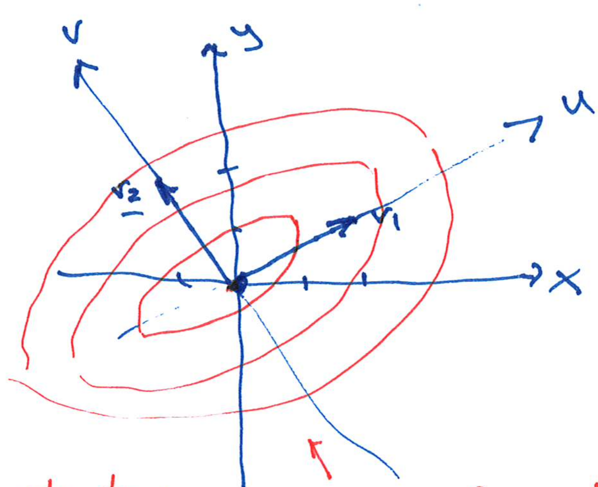
$$\begin{pmatrix} u \\ v \end{pmatrix} = P^T\cdot\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

P^T
$$\boxed{\begin{aligned} u &= \tfrac{1}{\sqrt{5}}(2x+y) \\ v &= \tfrac{1}{\sqrt{5}}(-x+2y) \end{aligned}}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = P\cdot\begin{pmatrix} u \\ v \end{pmatrix}$$

P
$$\boxed{\begin{aligned} x &= \tfrac{1}{\sqrt{5}}(2u-v) \\ y &= \tfrac{1}{\sqrt{5}}(u+2v) \end{aligned}}$$



Nivåkurver $f(x,y) = c$:

$5u^2 + 15v^2 = c$; (u,v)-koordinatsys.

$5u^2 + 15v^2 = c$

$c < 0$: ingen pkt. $c = 0$: $(x,y) = (0,0)$ pkt. $c > 0$: $\dfrac{u^2}{c/5} + \dfrac{v^2}{c/15} = 1$

Ellipse, sentr $(0,0)$, halvakser $a = \sqrt{c/5}$ $\; b = \sqrt{c/15}$