Oppgave:

Løs max $\int_0^1 1 - ty - \frac{1}{2}u^2 \, dt$ når $\begin{cases} y' = y + u \\ y(0) = 2 \\ y(1) = 3e - 1/e - 1 \end{cases}$

som et optimalt kontrollteori-problem.

Løsning.

$H = 1 - ty - \frac{1}{2}u^2 + p\cdot(y+u)$

(1) $H'_u = -u + p = 0$

(2) $-p' = H'_y = -t + p \Rightarrow p' = t - p$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \boxed{\begin{matrix} p = u \\ p' + p = t \\ y' = y + u \end{matrix}} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{fra maks-prinsippet} \\ \left.\begin{matrix} \\ \end{matrix}\right\} \text{fra problemet} \end{matrix}$$

$p' + p = t \quad |\cdot e^t$ (int. faktor)

$(pe^t)' = te^t$

$pe^t = \int te^t\,dt = te^t - e^t + C_1$ (delvis int.)

$p = \underline{t - 1 + C_1 e^{-t}} \qquad u = p = \underline{t - 1 + C_1 e^{-t}}$

$y' - y = u = t - 1 + C_1 e^{-t} \quad |\cdot e^{-t}$ (int. faktor)

$(ye^{-t})' = (t-1)e^{-t} + C_1 e^{-2t}$

$ye^{-t} = \int (t-1)e^{-t} + C_1 e^{-2t}\,dt$

$= (t-1)\cdot(-e^{-t}) - \int -e^{-t}\,dt + C_1\cdot\frac{1}{(-2)}e^{-2t} + C_2$

$ye^{-t} = -(t-1)e^{-t} - e^{-t} - \frac{1}{2}C_1 e^{-2t} + C_2 \quad |\cdot e^t$

$y = -(t-1) - 1 - \frac{1}{2}C_1 e^{-t} + C_2 e^t$

$y = \underline{-t - \frac{1}{2}C_1 e^{-t} + C_2 e^t}$

$$\left.\begin{aligned} y(0) &= -\tfrac{1}{2}C_1 + C_2 = 2 \\ y(1) &= -1 - \tfrac{1}{2}C_1\cdot\tfrac{1}{e} + C_2 e = 3e - 1/e - 1 \end{aligned}\right\} \begin{aligned} -\tfrac{1}{2}C_1 + C_2 &= 2 \\ -\tfrac{1}{2}C_1\cdot\tfrac{1}{e} + C_2\cdot e &= 3e - 1/e \end{aligned}$$

$\Rightarrow \underline{C_2 = 3,\ C_1 = 2}$

$y^* = \underline{-t - e^{-t} + 3e^t} \qquad u^* = \underline{t - 1 + 2e^{-t}}$

Ser at H er konkav i (y, u) siden Hesse-matrisen er

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$ som er neg. semidefin.

Derfor gir normal-løsn. $y^* = \underline{-t - e^{-t} + 3e^{t}}$ max.

Alt: Variansproblem

$y' = y + u \Rightarrow u = y' - y \Rightarrow$

$$\max \int_0^1 1 - ty - \frac{1}{2}u^2\,dt = \int_0^1 1 - ty - \frac{1}{2}(y'-y)^2\,dt \quad \text{når} \begin{cases} y(0) = 2 \\ y(1) = 3e - 1/e - 1 \end{cases}$$

Euler: $F'_y = -t - \frac{1}{2} \cdot 2(y'-y) \cdot (-1)$

$= -t + y' - y$

$F'_{y'} = -\frac{1}{2} \cdot 2(y'-y) \cdot 1 = -y' + y$

$\Rightarrow \frac{d}{dt}(F'_{y'}) = -y'' + y'$

Finne c_1, c_2 og sjekke at F er konkav i (y, y'): Som ovenfor

Euler: $-t + y' - y - (-y'' + y') = 0$

$$\boxed{y'' - y = t} \qquad y = y_h + y_p = \underline{C_1 e^{t} + C_2 e^{-t} - t}$$

y_h: $r^2 - 1 = 0$

$r = \pm 1 \Rightarrow y_h = \underline{C_1 e^{t} + C_2 e^{-t}}$

y_p: Utenfor pensum å gjøre det på denne måten, ville vært et hint: For eksempel: prøv $\underline{y_p = At + B}$

$\left.\begin{aligned} y &= At + B \\ y' &= A \\ y'' &= 0 \end{aligned}\right\}$ Setter inn:

$0 - (At + B) = t$ $\qquad y_p = \underline{-t}$

$-At - B = t$

$-A = 1$ $\quad -B = 0$ $\quad \Rightarrow A = -1$
(t-ledd) $\quad$ (konst. ledd) $\quad B = 0$