

① Row: $(3 \ -2 \ 4)$

column: $\begin{pmatrix} 6 \\ -1 \\ -3 \end{pmatrix}$

Dot product: $(\underline{3} \ \underline{-2} \ \underline{4}) \cdot \begin{pmatrix} \underline{6} \\ \underline{-1} \\ \underline{-3} \end{pmatrix}$

$= 18 + 2 - 12 = \underline{\underline{8}}$.

Order matters

$A \times B$ is allowed if and only if
 $(m \times n) (k \times l)$ $n = k$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 \\ 4 & 7 \end{bmatrix}$ $A \cdot B$
 $2 \times 2 \leftarrow \rightarrow 2 \times 2$ $m \times p \cdot p \times n$
 $A \cdot B = C$ $m \times n$

$AB = C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$
 2×2

②

$$C_{11} = (2 \ 1) \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} = 2 \cdot 1 + 1 \cdot 4 = \underline{\underline{6}}$$

$$C_{12} = (2 \ 1) \cdot \begin{pmatrix} -1 \\ 7 \end{pmatrix} = -2 + 1 \cdot 7 = \underline{5}$$

$$C_{21} = (-1 \ 3) \begin{pmatrix} 1 \\ 4 \end{pmatrix} = -1 \cdot 1 + 3 \cdot 4 = \underline{\underline{11}}$$

$$C_{22} = (-1 \ 3) \begin{pmatrix} -1 \\ 7 \end{pmatrix} = 1 + 21 = 22$$

$$A \cdot B = C = \begin{pmatrix} 6 & 5 \\ 11 & 22 \end{pmatrix}$$

Invertibility:

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A \cdot B = \begin{pmatrix} a+3 \cdot c & b+3 \cdot d \\ 1 \cdot a+3 \cdot c & 1 \cdot b+3 \cdot d \end{pmatrix} \\ \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Determinant

$$\det(A) = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

Example: $A = \begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix}$

$$\det(A) = 4 \cdot 2 - (-1) \cdot 3$$
$$= 8 + 3 = 11 \neq 0$$

Means A is invertible.

2nd ex: $A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$

$$\det(A) = 1 \cdot 3 - 1 \cdot 3 = 0$$

Not invertible = singular

④
Finding the inverse

$$A = \begin{bmatrix} \underline{4} & -1 \\ 3 & 2 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 - R_2$

$$\begin{bmatrix} 4-3=1 & -1-2=-3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ \underline{3} & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1-0 & 0-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3 \cdot R_1$

$$\begin{bmatrix} 1 & -3 \\ 3-3 \cdot 1 & 2-3 \cdot (-3) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & \underline{11} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 0 \cdot 3 \cdot 1 & 1-3(-1) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

$R_2 \rightarrow \frac{1}{11} R_2$

$$\begin{bmatrix} 1 & -3 \\ \frac{0}{11} & \frac{11}{11} \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & \underline{1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -\frac{3}{11} & \frac{4}{11} \end{bmatrix}$$

$R_1 \rightarrow R_1 + 3R_2$

$$\begin{bmatrix} 1+3 \cdot 0 & -3+3 \cdot 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1+3 \cdot \left(\frac{-3}{11}\right) & -1+3 \cdot \frac{4}{11} \\ -\frac{3}{11} & \frac{4}{11} \end{bmatrix}$$

~~1~~
2.

$$\begin{bmatrix} \frac{2}{11} & \frac{1}{11} \\ -\frac{3}{11} & \frac{4}{11} \end{bmatrix} = A^{-1}$$

~~det(A)~~ $\left(\frac{1}{11}\right) \cdot \begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix}$

$$= \frac{1}{\det(A)} \begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix}$$

5

Linear systems \rightarrow Matrices

$$2 \times 2: \begin{cases} 3x_1 - x_2 = 4 \\ x_1 + 2x_2 = -2 \end{cases}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{coefficient} & \text{constant} & \text{variable} \\ & \text{vector} & \text{vector} \\ A = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}, & \underline{b} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}, & \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{array}$$

$$\underline{x} = \begin{pmatrix} 2 \\ \cancel{1} \\ -1 \end{pmatrix}.$$

$$(\underline{b} \Rightarrow) A \underline{x} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ \cancel{1} \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \cdot \cancel{2} + (-1) \cdot (-1) \\ 1 \cdot 2 + 2 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A \underline{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \underline{b} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

So $\begin{pmatrix} x_1 = 2 \\ x_2 = -1 \end{pmatrix}$ does not solve above system.

⑥ Inverting 2x2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

determinant: $\det(A) = ad - bc \neq 0$

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example: $2x_1 - x_2 = 4$
 $-3x_1 + 2x_2 = 0$.

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$$A\underline{x} = \underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b}.$$

$$\det(A) = 2 \cdot 2 - 3 = 4 - 3 = \underline{1}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = 1 \cdot \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

$$\underline{x} = A^{-1}\underline{b} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \cdot 4 + 1 \cdot 0 \\ 3 \cdot 4 + 2 \cdot 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$ (checking)
 $2 \cdot 8 - 12 = 16 - 12 = \underline{4}$
 $-3 \cdot 8 + 2 \cdot 12 = \underline{0}$

(7)

Solving 3×3 :

$$6x_1 + 2x_2 + 6x_3 = 20$$

$$2x_1 + x_2 = 4$$

$$-4x_1 - 3x_2 + 9x_3 = 3.$$

$$A = \begin{pmatrix} 6 & 2 & 6 \\ 2 & 1 & 0 \\ -4 & -3 & 9 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 20 \\ 4 \\ 3 \end{pmatrix}.$$

$$A = \begin{pmatrix} \underline{6} & 2 & 6 \\ 2 & 1 & 0 \\ -4 & -3 & 9 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_1 \rightarrow \frac{1}{6}R_1$

$$\begin{pmatrix} \underline{\frac{6}{6}} & \frac{2}{6} & \frac{6}{6} \\ 2 & 1 & 0 \\ -4 & -3 & 9 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{3} & 1 \\ \underline{2} & 1 & 0 \\ \underline{-4} & -3 & 9 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 + 4R_1$

$$\begin{pmatrix} 1 & \frac{1}{3} & 1 \\ 0 & 1 - \frac{2}{3} & 0 - 2 \\ 0 & -3 + \frac{4}{3} & 9 + 4 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & -2 \\ 0 & \underline{\frac{-5}{3}} & 13 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{6} & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_2 \rightarrow 3 \cdot R_2$

$$\begin{pmatrix} 1 & \frac{1}{3} & 1 \\ 0 & 1 & -6 \\ 0 & -\frac{5}{3} & 13 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{6} & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_1 \rightarrow R_1 - \frac{1}{3}R_2$

$R_3 \rightarrow R_3 + \frac{5}{3}R_2$

$$\begin{pmatrix} 1 & 0 & 1 + 2 = 3 \\ 0 & 1 & -6 \\ 0 & 0 & 13 - \frac{6 \cdot 5}{3} \end{pmatrix} \quad \begin{matrix} 3 \\ 3 \\ 3 \end{matrix}$$

$$\begin{pmatrix} \frac{1}{6} & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

8

(cont...)

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & \underline{3} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \underline{3} \\ 0 & 1 & \underline{-6} \\ 0 & 0 & \underline{1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -1 & 0 \\ -1 & 3 & 0 \\ -1 & 5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -1 & 0 \\ -1 & 3 & 0 \\ -\frac{1}{3} & \frac{5}{3} & \frac{1}{3} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2}+1 & -6 & 0+1 \\ -1-2 & 3+0 & 0+2 \\ -\frac{1}{3} & \frac{5}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -6 & -1 \\ -3 & 3 & 2 \\ -\frac{1}{3} & \frac{5}{3} & \frac{1}{3} \end{pmatrix}$$

A^{-1}

Solve system: $\underline{x} = A^{-1} \cdot \underline{b}$

$$\underline{x} = \begin{pmatrix} \frac{3}{2} & -6 & -1 \\ -3 & 3 & 2 \\ -\frac{1}{3} & \frac{5}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 20 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \cdot 20 - 6 \cdot 4 - 1 \cdot 3 \\ -3 \cdot 20 + 3 \cdot 4 + 2 \cdot 3 \\ -\frac{1}{3} \cdot 20 + \frac{5}{3} \cdot 4 + \frac{1}{3} \cdot 3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$