FORK1003
Solutions for Exercises 2

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1 Matrices and Matrix Operations

1.1 Matrix Defined

Solution 1.1 (Matrix coordinates).

(a) 13

(b) 1

(c) 39

(d) -11

Explanation: In (i, j), i refers to the row and j refers to the column.

1.2 Addition and Scalar Multiplication

Solution 1.2.

(a) $\begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -6 & 6 \\ -5 & -8 & 5 \end{bmatrix}$

(c) Undefined

(d) $\begin{bmatrix} 3 & 0 \\ -24 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -14 & 15 \\ -10 & -24 & 17 \end{bmatrix}$

1.3 Matrix Multiplication

Solution 1.3.

(a) -1

(b) 29

(c) $a - b + 2c + 3e$

Solution 1.4.

(a) $\begin{bmatrix} 0 & -2 & 7 \\ -2 & 0 & -1 \end{bmatrix}$

(b) Undefined

(c) Undefined

(d) $\begin{bmatrix} 5 & -5 & -1 \\ 1 & 18 & -4 \end{bmatrix}$

(e) $\begin{bmatrix} 12 \\ -16 \\ -1 \end{bmatrix}$

(f) Undefined

(g) $\begin{bmatrix} -5 & -25 & 7 \end{bmatrix}$

(h) $\begin{bmatrix} 16 & 8 & -12 \\ -4 & -2 & 3 \\ 8 & 4 & -6 \end{bmatrix}$

(i) $\begin{bmatrix} 8 \end{bmatrix}$

1.5 Transpose

Solution 1.5.

(a) $\begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -1 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 12 \\ -4 & 0 \\ 7 & -4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -4 & 7 \\ 12 & 0 & -4 \end{bmatrix}$

1.7 Square Matrices

Solution 1.6. $A^2 = \begin{bmatrix} 13 & -20 \\ -5 & 8 \end{bmatrix}$

Solution 1.7.

(a) $\begin{bmatrix} 16 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}^2$

(b) $\begin{bmatrix} a_1^k & 0 & 0 \\ 0 & a_2^k & 0 \\ 0 & 0 & a_3^k \end{bmatrix}$

Explanation: When you multiply a diagonal matrix with itself, all you're doing is multiplying each diagonal entry with itself, and leaving the non-diagonal entries as zero.

2 Inverse Matrices

2.1 Briefly on Determinants

Solution 2.1.

(a) Determinant of A is 0, so not invertible.

(b) Determinant of B is -9, so invertible.

(c) Determinant of C is 0, so not invertible.

2.2 Finding the Inverse

Solution 2.2.

(a) $A^{-1} = \begin{bmatrix} -\dfrac{1}{2} & 1 \\ -\dfrac{3}{2} & 2 \end{bmatrix}$.

Possible order of row operations:

$$\begin{aligned} R1 &\to \frac{1}{4}R1 \\ R2 &\to R2 - 3R1 \\ R2 &\to 2R2 \\ R1 &\to R1 + \frac{1}{2}R2 \end{aligned}$$

(b) $B^{-1} = \begin{bmatrix} -40 & 16 & 7 \\ 13 & -5 & -2 \\ 5 & -2 & -1 \end{bmatrix}$.

Possible order of row operations:

$$\begin{aligned}
&R2 \to R2 - 3R1\\
&R3 \to R3 + R1\\
&R2 \to -R2\\
&R1 \to R1 - 2R2\\
&R3 \to R3 - 2R2\\
&R3 \to -R3\\
&R1 \to R1 - 7R3\\
&R2 \to R2 + 2R3
\end{aligned}$$

(c) $C^{-1} = \begin{bmatrix} 5/3 & -1/4 & 1/3 & -11/6 \\ -2/3 & 1/12 & 0 & 5/6 \\ -1/3 & 1/6 & 0 & -1/3 \\ 2/3 & -1/12 & 0 & 1/6 \end{bmatrix}$.

Possible order of row operations:

$$\begin{aligned}
&R1 \leftrightarrow R3\\
&R1 \to \frac{1}{3}R1\\
&R2 \to \frac{1}{2}R2\\
&R1 \to R1 - \frac{7}{3}R2\\
&R4 \to R4 - R2\\
&R1 \to R1 + 9R3\\
&R2 \to R2 - 4R3\\
&R4 \to -\frac{1}{2}R4\\
&R1 \to R1 - 11R4\\
&R2 \to R2 + 5R4\\
&R3 \to R3 - 2R4
\end{aligned}$$

2.3 Extra: Rearranging Matrix Equations

Solution 2.3.

$$D = B^{-1}AC^{-1} + 3I_n.$$

Explanation:

$$\begin{aligned} A &= B(D-3I_n)C \\ B^{-1}A &= (B^{-1}B)(D-3I_n)C \\ B^{-1}AC^{-1} &= I_n(D-3I_n)(CC^{-1}) \\ B^{-1}AC^{-1} &= (D-3I_n)I_n \\ B^{-1}AC^{-1} &= D-3I_n \\ B^{-1}AC^{-1}+3I_n &= D \end{aligned}$$

3 Linear Systems as Matrix Equations

Solution 3.1.

(a) $\begin{bmatrix} 0 & 1 & -3 & 1 \\ -1 & 0 & 1 & 6 \\ 9 & -7 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 13 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 & -1 \\ 3 & -4 & 5 & 0 \\ 0 & 15 & 2 & 0 \\ 1 & 1 & -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -20 \\ 0 \end{bmatrix}$

3.2 Solving Linear Systems Through Matrix Equations

Solution 3.2.

(a)

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 1/4 & 1/8 \\ -1/2 & 1/4 \end{bmatrix}, \qquad A^{-1}\underline{\mathbf{b}} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

so

$$x_1 = 0, \qquad x_2 = -2.$$

(b)

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & -4 & -4 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} -1 & 2 & 1/4 \\ -2 & 3 & 1/4 \\ 2 & -3 & -1/2 \end{bmatrix}, \qquad A^{-1}\underline{\mathbf{b}} = \begin{bmatrix} -5/2 \\ -11/2 \\ 6 \end{bmatrix}$$

so

$$x_1 = -5/2, \qquad x_2 = -11/2, \qquad x_3 = 6.$$

4 Linear Systems as Linear Combinations of Columns

Solution 4.1.

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 5 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}.$$

4.2 Linear Combinations

Solution 4.2.

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \underline{\mathbf{v}}_1 - \underline{\mathbf{v}}_2$

(b) $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \underline{\mathbf{v}}_1 - 2\underline{\mathbf{v}}_2 + 3\underline{\mathbf{v}}_3$

(c) Not a linear combination.

(d) $\begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix} = 4\underline{\mathbf{v}}_1 - 3\underline{\mathbf{v}}_2 + 2\underline{\mathbf{v}}_3$

(e) Not a linear combination.

(f) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \underline{\mathbf{v}}_1 - \underline{\mathbf{v}}_2 + \underline{\mathbf{v}}_3$

Hint: $\underline{\mathbf{v}}_1$ is the only vector with a non-zero entry in 2nd row. Similarly, $\underline{\mathbf{v}}_3$ is the only vector with a non-zero entry in 4th row, and $\underline{\mathbf{v}}_2$ is the only vector with a non-zero entry in 3rd row. From this you can derive the necessary coefficients.

4.4 Solving Linear Systems

Solution 4.3.

(a) These vectors span $\mathbb{R}^2$.

(b) These vectors span $\mathbb{R}^3$.

(c) These vectors do not span $\mathbb{R}^3$.

(d) These vectors span $\mathbb{R}^4$.

Solution 4.4. Coefficient matrix: $A = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$

Column vectors:

$$\underline{\mathbf{a}}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \qquad \underline{\mathbf{a}}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \underline{\mathbf{a}}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{4}\underline{\mathbf{a}}_1 - \frac{1}{4}\underline{\mathbf{a}}_2 + \frac{1}{4}\underline{\mathbf{a}}_3$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix} = -\frac{1}{4}\underline{\mathbf{a}}_1 + \frac{5}{4}\underline{\mathbf{a}}_2 + \frac{3}{4}\underline{\mathbf{a}}_3$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{1}{4}\underline{\mathbf{a}}_1 - \frac{1}{4}\underline{\mathbf{a}}_2 - \frac{3}{4}\underline{\mathbf{a}}_3$$

Since the column vectors span $\mathbb{R}^3$, we conclude that the linear system has a solution for every possible $\underline{\mathbf{b}}$.

