Plan

1. **Web page**: Link from It's L.
   - lecture notes (after each lecture)
   - problems for each lecture (from textbook)

2. **Solving linear systems of equations**

Ex:

$$\boxed{\begin{aligned} 5x + 7y &= 10 \\ -3x + 11y &= 18 \end{aligned}}$$

2×2 linear system

$\#$ equations $= \#$ variables

$$\boxed{\begin{aligned} x + y + z + w &= 10 \\ 2x + 3y \quad - w &= 12 \\ 5x - y + 3z - w &= 9 \end{aligned}}$$

3×4 linear system

linear equations = first order equations

Solution techniques:
- substitution methods
- elimination methods

(1) **Substitution**

(1) $5x + 7y = 10$

(2) $-3x + 11y = 18$

(1) $\frac{5x}{5} = \frac{10 - 7y}{5}$

$x = 2 - \frac{7}{5}y$

$= 2 - \frac{7}{5} \cdot \frac{30}{19} = 2 - \frac{42}{19}$

$= \frac{38-42}{19} = \underline{\underline{\frac{-4}{19}}}$

(2) $-3\left(2 - \frac{7}{5}y\right) + 11y = 18 \mid \cdot 5$

$-30 + 21y + 55y = 90$

$\frac{76y}{76} = \frac{120}{76}$ $\quad y = \frac{120}{76} = \underline{\underline{\frac{30}{19}}}$

Solution: $(x, y) = \underline{\underline{\left(-\frac{4}{19}, \frac{30}{19}\right)}}$

(2) Elimination:

$$5x + 7y = 10 \quad | \cdot 3$$
$$-3x + 11y = 18 \quad | \cdot 5$$

$\downarrow$

$$15x + 21y = 30$$
$$-15x + 55y = 90$$

$\downarrow$

$$15x + 21y = 30$$
$$76y = 120$$

$$y = \frac{120}{76} = \frac{60}{38} = \underline{\underline{\frac{30}{19}}}$$

$$15x + 21\left(\frac{30}{19}\right) = 30$$

$$15x = 30 - \frac{21 \cdot 30}{19}$$

$$x = 2 - \frac{21 \cdot \cancel{30}^{2}}{19 \cdot \cancel{15}}$$

$$= \frac{38}{19} - \frac{42}{19} = \underline{\underline{\frac{-4}{19}}}$$

③ <u>Key method:</u> Gaussian elimination

$\Rightarrow$ general method for solving any linear system

<u>Ex:</u>

$$\begin{aligned} x+y+z &= 1 \\ x+2y+4z &= 3 \\ x+3y+9z &= 7 \end{aligned}$$

3×3 linear system in std. form.

ⓐ

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 7 \end{array}\right) \cdot (-1)$$

<u>augmented matrix</u> of the linear system

$(-1 \ \ -1 \ \ -1 \ | \ -1)$

want to set 0 in these positions = under the pivot

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 1 & 3 & 9 & 7 \end{array}\right) \cdot (-1)$$

(b) Use <u>elementary row operations</u> on the <u>augmented matrix</u> until we have an <u>echelon form</u>

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 2 & 8 & 6 \end{array}\right) \cdot (-2)$$

want to get zero

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

echelon form

<u>Elementary row operations:</u>

i) switch two rows
ii) multiply a row with a number $c \neq 0$
iii) add a multiple of one row to another row

E.R.O. don't change the solutions of the system

Defn:

pivot = first non-zero number in a given row

echelon form: a matrix such that:

i) all zero rows are in the bottom of the matrix

ii) each pivot is further to the right than the pivots in the rows above.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

echelon form

$$\begin{aligned} x + y + z &= 1 \\ y + 3z &= 2 \\ 2z &= 2 \end{aligned}$$

(c) **Back substitution:**

$2z = 2 \quad \underline{z = 1}$

$y + 3z = 2 \quad y = 2 - 3z = 2 - 3\cdot 1 = \underline{-1}$

$x + y + z = 1 \quad x = 1 - y - z = 1 - (-1) - 1 = \underline{1}$

One solution: $(x, y, z) = \underline{\underline{(1, -1, 1)}}$

Overview: Gaussian elimination

linear system $\rightsquigarrow$ augmented matrix

} elementary row operations = Gaussian process

echelon form $\longrightarrow$ linear system

} back substitution

solutions

Note: – starting from any matrix, you can always get to an echelon form using elementary row operations

– but the echelon form is not unique

Ex:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 2 & 2 \end{array}\right) \cdot \tfrac{1}{2} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \; -3$$

echelon form

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \; -1 \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \; -1$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \qquad \begin{aligned} x &= 1 \\ y &= -1 \\ z &= 1 \end{aligned}$$

reduced echelon form
||
echelon form such that
(3) all pivots are 1
(4) all ~~pivots~~ entries over a pivot are 0

$\boxed{\text{Gauss-Jordan elimination}}$

note: the reduced echelon form is unique

Geometric picture:

Linear eqn: $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

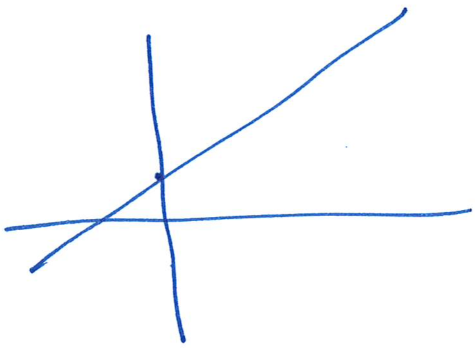
n=2: $ax + by = c$ straight line in $\mathbb{R}^2$

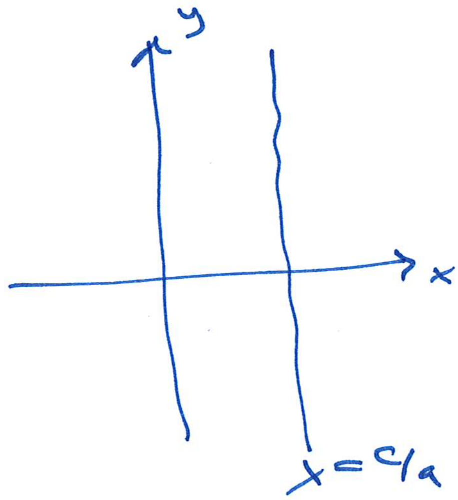
$b \neq 0$ $\qquad\qquad$ $b = 0$

$$\frac{by}{b} = \frac{c - ax}{b} \qquad\qquad ax = c$$

$$\boxed{y = \frac{c}{b} - \frac{a}{b} \cdot x} \qquad\qquad \boxed{x = c/a} \quad (a \neq 0)$$

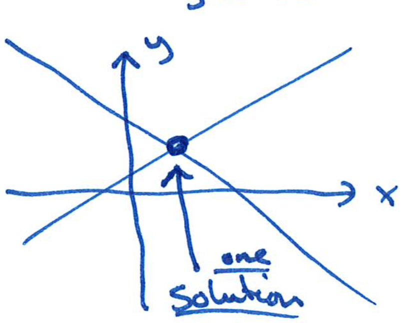
slope: $-a/b$
y-intercept: c/b

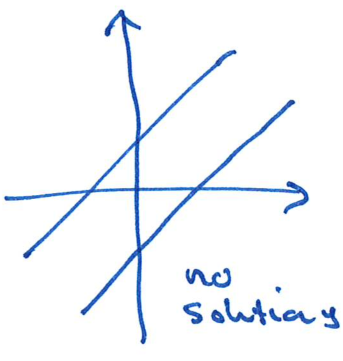


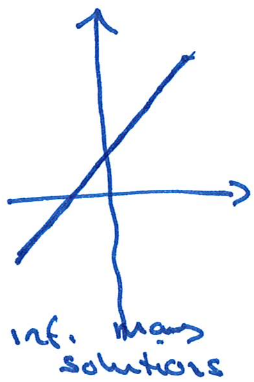


Ex: 2x2 linear system

$$5x + 3y = 12$$
$$-x + 7y = 10$$







Ex: $x+y+4z-w=7$
$2x-y+z+4w=3$
$x+4y+11z-7w=18$

$$\left(\begin{array}{cccc|c} 1 & 1 & 4 & -1 & 7 \\ 2 & -1 & 1 & 4 & 3 \\ 1 & 4 & 11 & -7 & 18 \end{array}\right) \begin{array}{l} -2 \\ -1 \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 4 & -1 & 7 \\ 0 & -3 & -7 & 6 & -11 \\ 0 & 3 & 7 & -6 & 11 \end{array}\right) \begin{array}{l} 1 \end{array}$$

$$\downarrow$$

$$\begin{array}{ccccc} x & y & z & w & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 1 & 4 & -1 & 7 \\ 0 & -3 & -7 & 6 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

echelon form

Solutions:

$(x,y,z,w) =$
$\left(\frac{10}{3}-\frac{5}{3}z-w,\ \frac{11}{3}-\frac{7}{3}z+2w,\ z,\ w\right)$

where z,w are free

$\Rightarrow$ infinitely many solutions

$x+y+4z-w=7$
$-3y-7z+6w=-11$
$~~0=0~~$

$$\frac{-3y}{-3} = \frac{-11+7z-6w}{-3}$$
$$y = \frac{11}{3} - \frac{7}{3}z + 2w$$

z,w: free variables
(no pivots in z- or w-column of echelon form)

x,y: basic variables
(which we can solve for)

$x = 7-y-4z+w$
$= 7-4z+w-\left(\frac{11}{3}-\frac{7}{3}z+2w\right)$
$= \frac{10}{3}-\frac{5}{3}z-w$

Ex:
$$\begin{aligned} x+2y-z&=7\\ 2x-y+3z&=11\\ x+7y-6z&=9 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 2 & -1 & 3 & 11 \\ 1 & 7 & -6 & 9 \end{array}\right) \begin{array}{l} \\ -2 \\ -1 \end{array}$$

$\downarrow$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & -5 & 5 & -3 \\ 0 & 5 & -5 & 2 \end{array}\right) \begin{array}{l} \\ \\ 1 \end{array}$$

$\downarrow$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & -5 & 5 & -3 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

echelon form

$$\begin{aligned} x+2y-z&=7\\ -5y+5z&=-3\\ 0&=-1 \quad \text{impossible} \end{aligned}$$

$\Downarrow$

no solutions

Result:

We consider any $m \times n$ linear system. Then we have:

i) there are <u>no solutions</u> $\Leftrightarrow$ there is a pivot position in the last column.

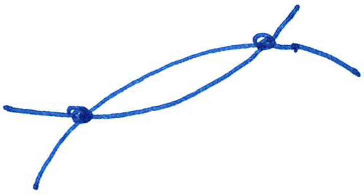
ii) we say that a variable is <u>basic</u> if there is a pivot position in the corresponding column, and <u>free</u> otherwise

there is <u>one unique solution</u> $\Leftrightarrow$ all variables are basic

there are <u>infinitely many solutions</u> $\Leftrightarrow$ there is at least one free variable

degrees of freedom = # free variables
= n − # pivot positions

In particular, any linear system has either <u>no solutions</u>, <u>one solution</u>, or <u>infinitely many solutions</u>.



<u>cannot have:</u>

<u>reason:</u> the solution of a linear equation ~~$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$~~

$$\boxed{a_1x_1 + a_2x_2 + \dots + a_nx_n = b}$$

<u>never curves</u>

<u>$n=3$:</u> a plane

<u>$n>3$:</u> a hyperplane