Plan

1 Optimization problems in one variable ← See Lecture 2.
2 Integration

② Integration.

Let $f(x)$ be a function (continuous fn. on an interval)

Defn: An antiderivative $F(x)$ of $f(x)$ is a fn. such that $F'(x) = f(x)$.

Ex: $f(x) = 6x - 2$

$$F(x) = 3x^2 - 2x + C$$
$$F'(x) = 6x - 2 = f(x)$$

$F(x) = \underline{3x^2 - 2x + C}$ is the general antiderivative of $f(x) = 6x - 2$

Fact: If $F(x)$ is one antiderivative of $f(x)$, then $F(x) + C$ is the general antiderivative.

Indefinite integral:

$$\int f(x)\, dx = F(x) + C$$

when $F'(x) = f(x)$.

$\int$ = integration sign

x is the integration variable

Integration rules: u, v expr. in x, c constant

1. $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \quad (n \neq -1)$ $\qquad (x^n)' = n x^{n-1}$
2. $\int u \pm v \, dx = \int u \, dx \pm \int v \, dx$ $\qquad (u \pm v)' = u' \pm v'$
3. $\int c \cdot u \, dx = c \cdot \int u \, dx$ $\qquad (c \cdot u)' = c \cdot u'$
4. $\int \frac{1}{x} \, dx = \ln|x| + C$ $\qquad (\ln x)' = 1/x$
5. $\int e^x \, dx = e^x + C$ $\qquad (e^x)' = e^x$

Ex: $\int x^3 - 3x + 2 \, dx = \frac{1}{4} x^4 - 3 \cdot \frac{1}{2} x^2 + 2x + C$

$$= \underline{\underline{\frac{1}{4} x^4 - \frac{3}{2} x^2 + 2x + C}}$$

$$\int \frac{1}{x^2} \, dx = \int x^{-2} \, dx = \frac{1}{-1} \cdot x^{-1} + C = \underline{\underline{-\frac{1}{x} + C}}$$

$$\int \frac{1 - x^2}{x} \, dx = \int \frac{1}{x} - \frac{x^2}{x} \, dx = \int \frac{1}{x} dx - \int x \, dx$$

$$= \int \frac{1}{x} dx - \frac{1}{2} x^2 + C$$

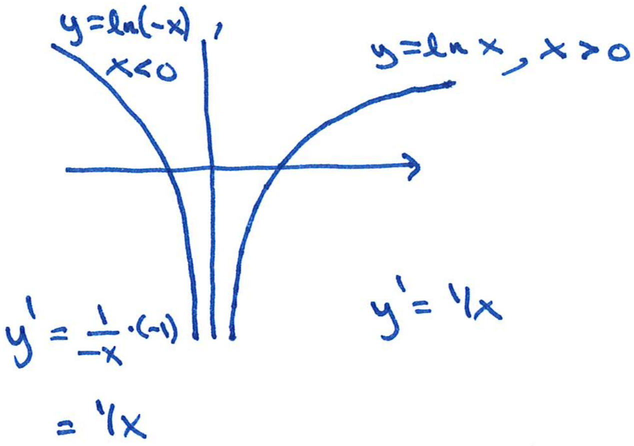
$$= \underline{\underline{\ln|x| - \frac{1}{2} x^2 + C}}$$

Ex: $\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{1}{3/2} x^{3/2} + C$

$n = 1/2 \to n+1 = 3/2$

$$= \underline{\underline{\frac{2}{3} x \cdot \sqrt{x} + C}}$$

Note: $\int \frac{1}{x}\,dx = \ln|x| + C$



$$f(x) = \begin{cases} \ln(x), & x>0 \\ \ln(-x), & x<0 \end{cases} = \ln|x|$$

Ex: $\int x \cdot \ln x\,dx$ ← integration by parts

$\int \frac{x}{1-x^2}\,dx$ ← partial fractions

$\int x\sqrt{1-x^2}\,dx$ ← substitution

(a) Integration by parts – to integrate products

$$\int u'v\,dx = uv - \int uv'\,dx$$

$(uv)' = u'v + uv'$

$\int (uv)'\,dx = \int u'v\,dx + \int uv'\,dx$

$uv = \int u'v\,dx + \int uv'\,dx$

Ex: $\int x \cdot \ln x\,dx$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}x^2 & v = \ln x \\ u' = x & v' = 1/x \end{array}}$$

$$= \frac{1}{2}x^2 \cdot \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\,dx$$

$$= \frac{1}{2}x^2 \cdot \ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\cdot\frac{1}{2}x^2 + C$$

$$= \underline{\underline{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}}$$

Ex: $\int \ln x\,dx = \int 1 \cdot \ln x\,dx$

$$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$$

$$= x \cdot \ln x - \int x \cdot \frac{1}{x}\,dx$$

$$= x \ln x - \int 1\,dx = \underline{\underline{x \ln x - x + C}}$$

(b) Substitution

Ex: $\int \frac{1}{1-x}\,dx = \int \frac{1}{u}\left(\frac{1}{-1}\right)du = -\int \frac{1}{u}\,du$

$\boxed{\begin{array}{l} u = 1-x \\ du = -1 \cdot dx \end{array}}$

$\boxed{du = u'\,dx}$

takes into consideration the derivative of the kernel

$dx = \frac{1}{-1}\,du$
$= -du$

$= -\ln|u| + C = \underline{\underline{-\ln|1-x| + C}}$

$\left(-\ln|1-x| + C\right)' = \left(\frac{1}{-1}\right)\left(\frac{1}{1-x} \cdot (-1)\right) = \frac{1}{1-x}$

Ex: $\int 6x\sqrt{1-x^2}\,dx = \int 6\cancel{x}\sqrt{u}\left(-\frac{1}{2\cancel{x}}\right)du$

$\boxed{\begin{array}{l} u = 1-x^2 \\ du = -2x\,dx \end{array}}$

$\Downarrow$

$dx = \frac{1}{-2x}\,du$

$= \int -3\sqrt{u}\,du = -3\int u^{1/2}\,du = -\cancel{3} \cdot \frac{2}{\cancel{3}}\,u^{3/2} + C$

$= -2u\sqrt{u} + C = \underline{\underline{-2(1-x^2)\sqrt{1-x^2} + C}}$

Ex:
$$\int \frac{1}{1-\sqrt{x}}\,dx = \int \frac{1}{1-u}\cdot 2\sqrt{x}\,du$$

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \end{aligned}}$$

$$\Updownarrow$$

$$dx = 2\sqrt{x}\,du$$

$$= \int \frac{1}{1-u}\cdot 2u\,du = \int \frac{2u}{1-u}\,du = \int -2 + \frac{2}{1-u}\,du$$

poly. div: $2u : (-u+1) = -2 + \frac{2}{1-u}$

$$\frac{2u-2}{2}$$

$$\int \frac{1}{1-\sqrt{x}}\,dx = \int \frac{1}{u}\cdot(-2\sqrt{x})\,du$$

$$\boxed{\begin{aligned} u &= 1-\sqrt{x} \\ du &= -\frac{1}{2\sqrt{x}}\,dx \end{aligned}}$$

$\Rightarrow \sqrt{x} = 1-u$

$\Rightarrow dx = -2\sqrt{x}\,du$

$$= \int \frac{-2(1-u)}{u}\,du = \int \frac{2u-2}{u}\,du$$

$$= \int 2 - \frac{2}{u}\,du = 2u - 2\ln|u| + C$$

$$= \underline{\underline{2(1-\sqrt{x}) - 2\ln|1-\sqrt{x}| + C}}$$

(c) Partial fractions

$$\int \frac{1}{ax+b}\,dx = \int \frac{1}{u}\left(\frac{1}{a}du\right) = \frac{1}{a}\int \frac{1}{u}\,du$$

$$\boxed{\begin{aligned} u &= ax+b \\ du &= a \cdot dx \end{aligned}}$$

$$= \frac{1}{a}\ln|u| + C = \underline{\frac{1}{a}\ln|ax+b| + C}$$

Ex: $\displaystyle\int \frac{2}{1-x^2}\,dx \leftarrow$ partial fractions

$$\frac{2}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \quad \Big| \cdot (1-x^2)$$ find A and B

$$\boxed{\begin{aligned} u &= 1-x^2 \\ &= (1-x)(1+x) \end{aligned}}$$

$$\frac{2}{\cancel{1-x^2}} \cdot \cancel{(1-x^2)} = \frac{A}{\cancel{1-x}} \cdot \cancel{(1-x)}(1+x) + \frac{B}{\cancel{1+x}} \cdot (1-x)\cancel{(1+x)}$$

$$\begin{aligned} 2 &= A(1+x) + B(1-x) \\ &= A + Ax + B - Bx \end{aligned}$$

$$\boxed{\frac{1}{1-x} + \frac{1}{1+x} = \frac{2}{1-x^2}}$$

$$2 = (A-B)x + (A+B)$$

compare coefficients:

$$\begin{aligned} A - B &= 0 \quad \Rightarrow A = B \\ A + B &= 2 \quad\quad 2A = 2 \\ & \quad\quad \underline{A=1} \quad \underline{B=1} \end{aligned}$$

$$\int \frac{2}{1-x^2}\,dx = \int \frac{1}{1-x} + \frac{1}{1+x}\,dx = \underline{-\ln|1-x| + \ln|1+x| + C}$$

$$= \underline{\underline{\ln\frac{|1+x|}{|1-x|} + C}}$$

$\leftarrow \ln a - \ln b = \ln(a/b)$

Ex: $\int \frac{x^2}{1-x^2}\,dx = \int -1 + \frac{1}{1-x^2}\,dx = -x + \underbrace{\int \frac{1}{1-x^2}\,dx}$

poly. div: $x^2 : (-x^2+1) = -1 + \frac{1}{1-x^2}$

$$\begin{array}{l} x^2 - 1 \\ \hline \quad 1 \end{array}$$

$$= -x + \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$

for partial fractions to work on

$$\frac{p(x)}{q(x)}$$

we need ~~that~~ $\deg p < \deg q$