Plan

① Introduction : Eivind Eriksen
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② Solving linear systems of equations

Ex: $5x + 7y = 8$
$3x - 8y = 17$

2×2 linear system
#eqns. (linear) #variables

$x + y + z + w = 10$
$2x + 3y - w = 12$
$5x - y + 3z - w = 9$

3×4 linear system

Solution methods
- substitution methods
- elimination methods

i) Substitution

$5x + 7y = 8$
$3x - 8y = 17$

$\frac{7y}{7} = \frac{8-5x}{7}$

$y = \frac{8}{7} - \frac{5}{7}x$

$3x - 8\left(\frac{8}{7} - \frac{5}{7}x\right) = 17$ $\cdot 7$

$21x - 8(8-5x) = 119$

$\frac{61x}{61} = \frac{183}{61}$ $\underline{\underline{x = 3}}$

$y = \frac{8}{7} - \frac{15}{7} = \underline{\underline{-1}}$

Solution:
$(x,y) = \underline{\underline{(3,-1)}}$

ii) Elimination

$$\left(\begin{array}{cc|c} 5 & 7 & 8 \\ 3 & -8 & 17 \end{array}\right) \begin{array}{l} \cdot 3 \\ \cdot 5 \end{array}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ x & y \\ \text{col.} & \text{col.} \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{cc|c} 15 & 21 & 24 \\ 15 & -40 & 85 \end{array}\right) -1$$

$$\downarrow$$

$$\left(\begin{array}{cc|c} 15 & 21 & 24 \\ 0 & -61 & 61 \end{array}\right)$$

$$5x + 7y = 8 \quad | \cdot 3$$
$$3x - 8y = 17 \quad | \cdot 5$$

$$\downarrow$$

$$\text{I} \quad 15x + 21y = 24$$
$$\text{II} \quad 15x - 40y = 85$$

$$\downarrow$$

$$\text{I} \quad 15x + 21y = 24$$
$$\text{II} - \text{I} \quad -61y = 61$$

$$y = \frac{61}{-61} = -1$$

$$15x - 21 = 24$$

$$\frac{15x}{15} = \frac{45}{15}$$

$$x = 3$$

Solution: $(x, y) = (3, -1)$

③ Key method: Gaussian elimination

Ex:

$$\begin{aligned} x+y+z&=3 \\ x+2y+4z&=7 \\ 2x+3y+6z&=12 \end{aligned} \quad \rightsquigarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 2 & 3 & 6 & 12 \end{array}\right)$$

3x3 lin. system (in std. form) — augmented matrix

Defn. Elementary row operations

i) Switch two rows

ii) Multiply a row by $c \neq 0$

iii) Add a multiple of one row to another row

Defn. Pivots

The first non-zero entry in a row is called a *pivot*.

Ex:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 2 & 3 & 6 & 12 \end{array}\right) \begin{matrix} -1 \\ \\ \\ \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & 3 & 6 & 12 \end{array}\right) \begin{matrix} \\ -2 \\ \\ \end{matrix} \to$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 4 & 6 \end{array}\right) \begin{matrix} \\ \\ -1 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

echelon form

<u>Defn</u> A matrix is in <u>echelon form</u> if the following conditions hold:

i) If there are rows of zeros, they should be in the bottom of the matrix

ii) Each pivot is further to the right than pivots in the rows above.

<u>Procedure:</u>

- start in the upper left corner
- if it is zero: switch rows to get it to be non-zero

$\Rightarrow$ first pivot in the upper left corner

- use elementary row operations of type III, adding multiples of row <u>1</u> (using the first pivot), to get zeros under the first pivot

$$\left(\begin{array}{ccccc|c} * & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ \vdots & \vdots & \vdots & & & \\ 0 & * & * & & * & * \end{array}\right)$$

- repeat the process with the <u>red submatrix</u>

<u>Method so far:</u> ① Write down the augmented matrix

② use elementary row operations until you have an echelon form

Ex:
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right)$$
echelon form

$$\begin{aligned} \underline{x} + y + z &= 3 \\ \underline{y} + 3z &= 4 \\ \underline{z} &= 2 \end{aligned}$$

Back substitution: $\underline{z = 2}$

$$\cancel{z}\, y + 3z = 4$$
$$y + 3 \cdot 2 = 4 \qquad y = \underline{-2}$$

Solution:

$(x,y,z) = \underline{\underline{(3,-2,2)}}$

$$x + y + z = 3$$
$$x - 2 + 2 = 3$$
$$\underline{x = 3}$$

Remarks:

(1) Elementary row operations preserves the solutions of linear systems.

(2) Any matrix can be transformed into an echelon form using elementary row operations.

(3) There is not a unique echelon form, but the pivot positions are unique.

(4) The pivot positions determine the number of solutions in a linear system.

Ex:

$$\begin{aligned} x+y+z&=3\\ x+2y+4z&=7\\ 2x+3y+5z&=12 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 1 & 2 & 4 & 7\\ 2 & 3 & 5 & 12 \end{array}\right) \begin{array}{l} \\ -1 \\ -2 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 1 & 3 & 4\\ 0 & 1 & 3 & 6 \end{array}\right) -1 \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 1 & 3 & 4\\ 0 & 0 & 0 & 2 \end{array}\right)$$

echelon form

$$\begin{aligned} x+y+z&=3\\ y+3z&=4\\ 0&=2 \end{aligned}$$

no solutions

Remark: Any linear system with a pivot position in the last column has no solutions.

Ex:

$$\begin{aligned} x+y+z+w&=4\\ 2x-y+z+2w&=4\\ x+4y-z+w&=5 \end{aligned}$$

3×4 linear system

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4\\ 2 & -1 & 1 & 2 & 4\\ 1 & 4 & -1 & 1 & 5 \end{array}\right) \begin{array}{l} \\ -2 \\ -1 \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4\\ 0 & -3 & -1 & 0 & -4\\ 0 & 3 & -2 & 0 & 1 \end{array}\right) 1$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4\\ 0 & -3 & -1 & 0 & -4\\ 0 & 0 & -3 & 0 & -3 \end{array}\right)$$

echelon form

$$\begin{aligned} x+y+z+w&=4\\ -3y-z&=-4\\ -3z&=-3 \end{aligned}$$

$-3z = -3 \quad \underline{z=1}$

$-3y - z = -4$

$-3y - 1 = -4 \quad -3y = -3 \quad \underline{y = 1}$

$x + y + z + w = 4$

$x + 1 + 1 + w = 4$

$x + 2 + w = 4 \quad x = \underline{2 - w}$

Solutions: $(x, y, z, w) = \underline{(2-w, 1, 1, w)}$

where w is a free variable

$\Rightarrow$ infinitely many solutions

In general: If you have an echelon form without a pivot in the last column:

Basic variable: There is a pivot position in the corresponding col.

Free variable: Otherwise

In the example:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & -3 & -1 & 0 & -4 \\ 0 & 0 & -3 & 0 & -3 \end{array}\right)$$
$x \quad y \quad z \quad w$

x, y, z: basic (can solve for these)

w : free

Result:

Let us consider any $m \times n$ linear system. Then we have:

i) there are no solutions $\Leftrightarrow$ there is a pivot position in the last column.

ii) If there is no pivot position in the last column:

a) there is one unique solution $\Leftrightarrow$ all variables are basic

b) there are infinitely many solutions $\Leftrightarrow$ there is at least one free variable

In particular:

Any linear system has either

- no solutions
- one solution (unique)
- inf. many solutions

If there are infinitely many solutions

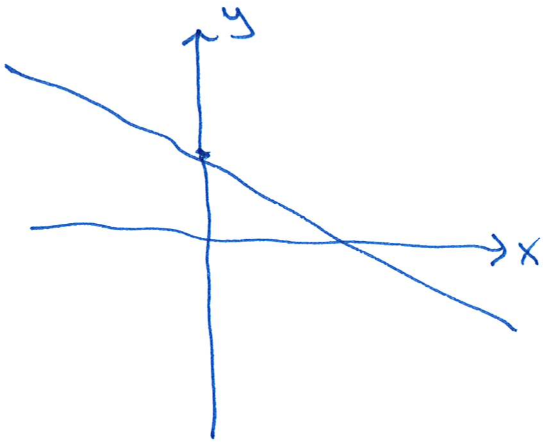
Defn: The number of degrees of freedom $= \#$ free variables.

Geometric picture

$n=2$: $5x+7y=8$

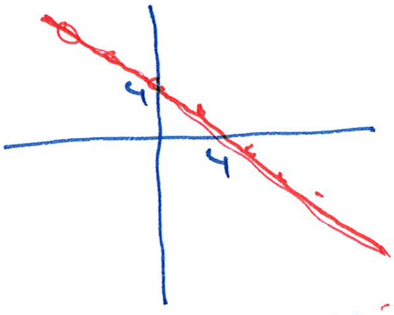
$$7y = 8-5x$$

$$y = \frac{8}{7} - \frac{5}{7}x$$



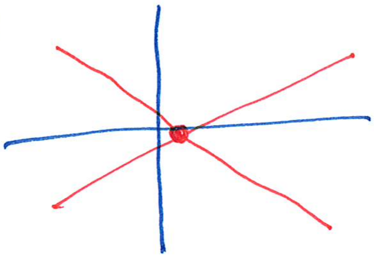
linear eqn = str. line

2x2 lin. system:

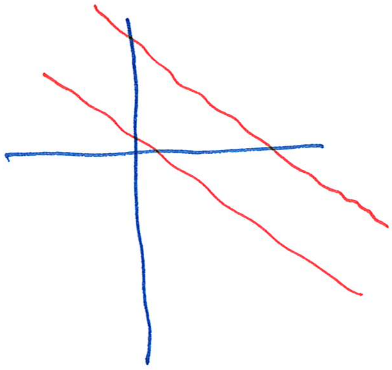


inf. many solutions

one. degree of freedom



one solution



no solutions

Ex: $x+y=4$

$2x+2y=8$

y free, $x=4-y$

Note:

degrees of freedom = dimension of the solution space.