Plan

1 Integration

(1) Integration

$f(x)$: function in one variable

Defn: A function $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$

Ex: $f(x) = 2x$ $\rightarrow$ $F(x) = x^2$ is an antiderivative of $f(x) = 2x$

$F(x) = x^2 + C$ is the general antiderivative of $f(x) = 2x$.

(C is a constant)

Indefinite integral:

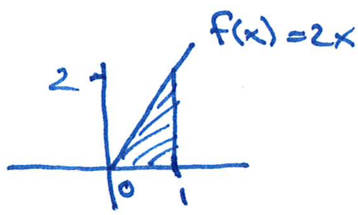
Ex: $\int 2x \, dx = x^2 + C$ ← the general antiderivative of $f(x) = 2x$ w.r.t. x

integration sign

$f(x)$

means that x is the integration variable

integration constant



Definite integral:

Ex: $\int_0^1 2x \, dx = \left[x^2 + C \right]_0^1 = (1^2 + C) - (0^2 + C) = \underline{\underline{1}}$

Ex: $y' = 2x$ differential equation

$$y = \int 2x\,dx = x^2 + C$$

How to compute indefinite integrals

Ex: $\int x^2 - 3x + 4\,dx = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 4x + C$

Integration rules:

(1) $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$

(2) $\int \frac{1}{x}\,dx = \ln|x| + C$

(3) $\int u(x) \pm v(x)\,dx = \int u(x)\,dx \pm \int v(x)\,dx$

(4) $\int c \cdot u(x)\,dx = c \cdot \int u(x)\,dx$

(5) $\int e^x\,dx = e^x + C$

$\int a^x\,dx = a^x \cdot \frac{1}{\ln(a)} + C$

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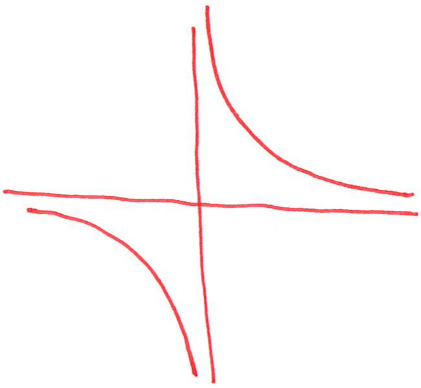
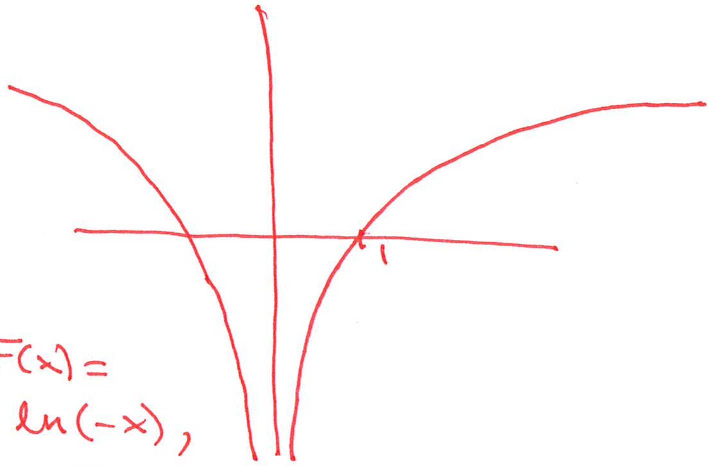
Explanation: $\int \frac{1}{x}\,dx = \ln|x| + C$



$f(x) = 1/x\,,\ x \neq 0$



$F(x) = \ln(-x)\,,\ x < 0$

$F(x) = \ln(x)\,,\ x > 0$

$F'(x) = \frac{1}{-x}\cdot(-1) = 1/x$

$F'(x) = 1/x$

$$F(x) = \begin{cases} \ln(x), & x > 0 \\ \ln(-x), & x < 0 \end{cases} = \ln|x|$$

Ex: $\int x^3 - 5x + 1\,dx = \frac{1}{4}x^4 - 5\cdot\frac{1}{2}x^2 + x + C$

$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + C = \frac{2}{3}x\cdot\sqrt{x} + C$

$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{1}{-1}\cdot x^{-1} + C = -1/x + C$

$\int x\cdot e^x\,dx = ?$ $\int \sqrt{1-x}\,dx = ?$

$\int \frac{x+1}{1-x^2}\,dx = ?$

Integration by parts — "to integrate products"

$$\boxed{\int u' \cdot v \, dx = u \cdot v - \int u v' \, dx}$$

$(uv)' = u'v + uv'$
$uv = \int u'v \, dx + \int uv' \, dx$

Ex: $\cancel{\int x \cdot e^x \, dx = \frac{1}{2}x^2 \cdot e^x - \int \frac{1}{2}x^2 \cdot e^x \, dx}$

$\cancel{\boxed{\begin{array}{ll} u = \frac{1}{2}x^2 & v = e^x \\ u' = x & v' = e^x \end{array}}}$

$$\int x \cdot e^x \, dx = x e^x - \int e^x \cdot 1 \, dx = x e^x - \int e^x \, dx$$

$$\boxed{\begin{array}{ll} u = e^x & v = x \\ u' = e^x & v' = 1 \end{array}}$$

$$= \underline{\underline{x e^x - e^x + C}}$$

Ex: $\int x \cdot \ln x \, dx = \frac{1}{2}x^2 \cdot \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x} \, dx$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}x^2 & v = \ln x \\ u' = x & v' = 1/x \end{array}}$$

$$= \frac{1}{2}x^2 \ln x - \int \frac{1}{2} x \, dx = \frac{1}{2}x^2 \ln x - \frac{1}{2} \cdot \frac{1}{2}x^2 + C$$

$$= \underline{\underline{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}}$$

Ex: $\int \ln x \, dx = \int 1 \cdot \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx$

$$\boxed{\begin{matrix} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{matrix}}$$

$$= x \ln x - \int 1 \, dx = \underline{\underline{x \ln x - x + C}}$$

Substitution: $e^{1-2x} = e^u, \quad u = 1-2x$

Ex: $\int e^{1-2x} \, dx = \int e^u \cdot dx = \int e^u \cdot (-\tfrac{1}{2}) \, du$

$$\boxed{\begin{matrix} u = 1-2x \\ du = u' \cdot dx \end{matrix}}$$

$$du = -2dx$$
$$-\tfrac{1}{2} du = dx$$

$$= -\frac{1}{2} \int e^u \, du = \underline{-\frac{1}{2} e^u + C} = \underline{\underline{-\frac{1}{2} e^{1-2x} + C}}$$

Formula: $\boxed{du = u' \cdot dx}$ $\qquad \frac{1}{u'} du = dx$

Ex: $\int x \cdot \sqrt{1-x^2} \, dx = \int \cancel{x} \cdot \sqrt{u} \, \frac{1}{-2\cancel{x}} \, du$

$$\boxed{\begin{matrix} u = 1-x^2 \\ du = -2x \, dx \end{matrix}}$$

$$= \int -\frac{1}{2} \sqrt{u} \, du = -\frac{1}{2} \int u^{1/2} \, du = -\frac{1}{\cancel{2}} \frac{\cancel{2}}{3} u^{3/2} + C = \underline{-\frac{1}{3} (1-x^2)^{3/2} + C}$$
$$= \underline{-\frac{1}{3} (1-x^2) \cdot \sqrt{1-x^2} + C}$$

Partial fractions

A rational function: $f(x) = \dfrac{p(x)}{q(x)}$ ← polynomials

Ex: $\displaystyle\int \frac{3}{1+x}\,dx = \int \frac{3}{u}\,du = 3\ln|u| + C$

$\boxed{\begin{aligned} u &= 1+x \\ du &= 1\cdot dx \end{aligned}}$

$= \underline{\underline{3\ln|1+x| + C}}$

$$\int \frac{x^2}{1+x}\,dx = \int \frac{x^2}{u}\,du = \int \frac{(u-1)^2}{u}\,du$$

$\boxed{\begin{aligned} u &= 1+x \\ du &= 1\cdot dx \end{aligned}}$ $x = u-1$

$$= \int \frac{u^2-2u+1}{u}\,du = \int u - 2 + \frac{1}{u}\,du$$

$$= \frac{1}{2}u^2 - 2u + \ln|u| + C$$

$$= \underline{\frac{1}{2}(1+x)^2 - 2(1+x) + \ln|1+x| + C}$$

$$= \frac{1}{2}(1+2x+x^2) - 2(1+x) + \ldots$$

$$= \frac{1}{2}x^2 - x + \ldots$$

Polynomial div:

$x^2 : x+1 = x - 1 + \frac{1}{1+x}$

$-(x^2+x)$

$-x$

$-(-x-1)$

1

$$\int \frac{x^2}{1+x}\,dx = \int x - 1 + \frac{1}{1+x}\,dx$$

$$= \underline{\frac{1}{2}x^2 - x + \ln|1+x| + C}$$

Ex: $\int \frac{2x+1}{1-x^2}\, dx$ smaller deg. deg. 2

Partial fractions: $\frac{2x+1}{1-x^2} = \frac{\overset{3/2}{A}}{1-x} + \frac{\overset{-1/2}{B}}{1+x} \quad | \cdot (1-x)(1+x)$

$1-x^2 = (1-x)\cdot(1+x)$

$2x+1 = A(1+x) + B(1-x)$

~~$x=-1$: $0 = A\cdot 0 + B\cdot 2 \quad 2B=0 \quad B=0$~~

~~$x=1$: $2 = A\cdot 2 + B\cdot 0 \quad 2A=2 \quad A=1$~~

$x=-1$: $-1 = A\cdot 0 + B\cdot 2 \quad 2B=-1 \quad B=-1/2$

$x=1$: $3 = A\cdot 2 + B\cdot 0 \quad 2A=3 \quad A=3/2$

$$\int \frac{2x+1}{1-x^2}\, dx = \int \frac{3/2}{1-x} - \frac{1/2}{1+x}\, dx$$

$$= -\frac{3}{2}\ln|1-x| - \frac{1}{2}\ln|1+x| + C$$

$u=1-x$ $\qquad$ $u=1+x$

$$\int \frac{3/2}{1-x}\, dx = 3/2 \int \frac{1}{u}\left(\frac{1}{-1}\right) du$$

$\boxed{\begin{array}{l} u=1-x \\ du=-dx \end{array}}$

$$= -\frac{3}{2}\ln|u| + C$$

$$= -\frac{3}{2}\ln|1-x| + C$$

Ex: (difficult)

$$\int \frac{1}{1+\sqrt{x}}\,dx$$