Plan

1 Key Method: Determinants
2 Linear systems, inverse matrices and determinants

Result: A $n \times n$ matrix

A is invertible (A^{-1} exists) $\Leftrightarrow$ $|A| \neq 0$

If $|A| \neq 0$, then

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A)$$

$|A| = \det(A)$ determinant of A

① Determinants

$$\underset{n \times n \text{ matrix}}{A} \rightsquigarrow \det(A) = |A|$$

determinant of A

the determinant is a number

n=2: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underline{ad - bc}$

n=1: $A = (a)$ $\det(A) = |A| = a$

Ex: $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 2 \cdot 2 - (-1) \cdot (-1)$

$$= 4 - 1 = \underline{3}$$

How to compute determinants ($n \geq 3$)

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{matrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{matrix}$

Method I: Works for 3x3 matrices
not the best method
does not work for n>3

$$|A| = 1 \cdot 2 \cdot 9 + 1 \cdot 4 \cdot 1 + 1 \cdot 1 \cdot 3 - 1 \cdot 2 \cdot 1 - 3 \cdot 4 \cdot 1 - 9 \cdot 1 \cdot 1$$

$$= 18 + 4 + 3 - 2 - 12 - 9 = \underline{\underline{2}}$$

Method 2: Cofactor expansion along a row or column – works for any square matrix

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$

$|A| = 1 \cdot C_{11} + 1 \cdot C_{12} + 1 \cdot C_{13}$

$= +1 \cdot M_{11} - 1 \cdot M_{12} + 1 \cdot M_{13}$

$= +1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$

$= 1 \cdot 6 - 1 \cdot 5 + 1 \cdot 1$

$= 6 - 5 + 1 = \underline{\underline{2}}$

C_{ij}: cofactor of A in position (i,j) = row i, col. j

$C_{ij} = \underbrace{(-1)^{i+j}}_{\substack{\text{sign} \\ (\pm 1)}} \cdot \underbrace{M_{ij}}_{\substack{\text{minor} \\ (\text{number})}}$

Signs: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

minors: M_{ij} is the determinant of the submatrix you obtain by deleting row i, col j.

$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ $\quad M_{11} = \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} = 2 \cdot 9 - 3 \cdot 4 = 6$

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ $\quad M_{12} = \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} = 9 - 4 = 5$

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ $\quad M_{13} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3 - 2 = 1$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = -1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} - 3 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}$$

$$= -1 \cdot 5 + 2 \cdot 8 - 3 \cdot 3 = -5 + 16 - 9 = \underline{\underline{2}} \leftarrow$$

A is inv. (A^{-1} exists)

Result: The cofactor expansion of A along any row or column gives the same result, det(A).

A $n \times n$ matrix

Ex:

$$\begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix} = +1 \cdot \begin{vmatrix} 4 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \cancel{0} + \cancel{0} - (-1) \cdot \begin{vmatrix} 0 & 0 & -1 \\ 4 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix}$$

$$= +1 \left(+\cancel{0} - \cancel{0} + 1 \cdot \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} \right) + 1 \cdot \left(+\cancel{0} - \cancel{0} - 1 \cdot \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} \right)$$

$$= \underset{7}{\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix}} - \underset{7}{\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix}} = \underline{\underline{0}} \leftarrow$$

the matrix is not invertible (A^{-1} does not exist)

In general:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

$n \times n$ matrix

formula for $|A|$: complicated

$n! = n \cdot (n-1)(n-2) \cdots 2 \cdot 1$ terms of degree n

How to compute A^{-1}: If $|A| \neq 0$, then $A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A)$

Defn: The adjuncated matrix $\mathrm{adj}(A)$ of an $n \times n$ matrix A is given by

$$\mathrm{adj}(A) = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T$$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ $\quad |A| = 1\cdot 6 + 1\cdot(-5 + 1 \cdot 1 = \underline{2}$

$C_{11} = +\begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} = 6 \qquad C_{12} = -\begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} = -5 \qquad C_{13} = +1$

$C_{21} = -6 \qquad C_{22} = +8 \qquad C_{23} = -2$

$C_{31} = +2 \qquad C_{32} = -3 \qquad C_{33} = +1$

$$A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A) = \frac{1}{2}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}^T = \underline{\underline{\frac{1}{2}\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}}}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}}_{A} \cdot \underbrace{\frac{1}{2}\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}}_{A^{-1}} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \underset{|A|}{2} \end{pmatrix}$$

Alternative method for computing determinants:

Using Gaussian process.

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$

Defn: A square matrix A is upper triangular if all entries under the main diagonal are zero.

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = +1 \cdot \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = +1\,(+1 \cdot 2) = 1 \cdot 1 \cdot 2 = \underline{\underline{2}}$$

Result: * If A is an upper triangular matrix, then $|A|$ is the product of the diagonal entries.
* An echelon form (square) is upper triangular.

Ex: $$\begin{vmatrix} 7 & 14 & -1 & 3 \\ 0 & \sqrt{2} & 1 & 0 \\ 0 & 0 & -\sqrt{2} & 1 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 7 \cdot \sqrt{2} \cdot (-\sqrt{2}) \cdot 3 = \underline{\underline{-42}}$$

Ex: $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 4 \\ 0 & 0 & -1 \end{pmatrix}$ echelon form, $|A| = 1 \cdot 7 \cdot (-1) = -7$

$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$ echelon form, $|A| = 1 \cdot 0 \cdot 0 = 0$

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix} = E \qquad |E| = 1 \cdot 1 \cdot 2 = \underline{2}$$

$$|A| = \underline{\underline{2}}$$

echelon form

Result: $A \rightarrow B$ elementary row operation

i) Add a multiple of one row to another row: $|B| = |A|$

ii) Multiply a row by $c \neq 0$: $|B| = c \cdot |A|$

iii) Switch two rows: $|B| = -|A|$

Ex:

$$\begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = \underline{\underline{0}}$$

Formulas that are useful:

i) $|A \cdot B| = |A| \cdot |B|$

ii) $|A^{-1}| = 1/|A|$ $\qquad A^{-1} \cdot A = I \Rightarrow |A^{-1}| \cdot |A| = 1$

iii) $|A^T| = |A|$

iv) $(A \cdot B)^T = B^T \cdot A^T$

v) $(AB)^{-1} = B^{-1} \cdot A^{-1}$

<u>Result:</u> The determinant of A is zero if either

i) A has a zero row

ii) A has two equal rows

iii) A has one row that is a multiple of another row

iv) A has one row that is a sum of multiples of other rows ("linear combination of other rows")

The same applies to columns.

<u>Ex:</u>

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & 3 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{|A| = 0}$$

$$B = \begin{pmatrix} 1 & 4 & 7 \\ 3 & 1 & 9 \\ 2 & 8 & 14 \end{pmatrix} \xrightarrow{-2} \begin{pmatrix} 1 & 4 & 7 \\ 3 & 1 & 9 \\ 0 & 0 & 0 \end{pmatrix} \qquad |B| = 0$$

R(1) + R(3) = R(4)

$$C = \begin{pmatrix} 1 & 7 & 3 & 5 \\ 2 & 5 & 1 & 4 \\ 3 & -4 & 0 & 1 \\ 4 & 3 & 3 & 6 \end{pmatrix} \xrightarrow{-1,\,-1} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad |C| = 0$$

(2) Connection between determinants and linear systems

Linear $n\times n$ system in matrix form: $A\underline{x}=\underline{b}$

$\Rightarrow$ A $n\times n$-matrix $\Rightarrow$ can compute $|A|$

Ex:

$$x+y+z=3$$
$$x+2y+4z=7$$
$$x+3y+9z=13$$

3×3 lin. sys.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3 \\ 7 \\ 13 \end{pmatrix}$$

$$|A|=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix}=2$$

Ex:

$$x+y+z+w=4$$
$$x-y+z+w=2$$
$$x+y-z-w=0$$

3×4 lin. sys.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}=\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

$|A|$ not defined

<u>Result:</u> If $A\underline{x}=\underline{b}$ is an $n\times n$-linear system, then:

$|A|\neq 0$: one unique solution of $A\underline{x}=\underline{b}$

$(\underline{x}=A^{-1}\cdot\underline{b})$

$|A|=0$: no solutions or inf. many solutions of $A\underline{x}=\underline{b}$

$$A = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$E = \begin{pmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 0 & 0 & \cdot \end{pmatrix}$$

$|E| \neq 0 \iff |A| \neq 0$

all variables are basic

one unique sol.

$$A = \left(\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}\right) \left| \, \underline{b} \right.$$

$$E = \left(\begin{matrix} \cdot & \cdot & \cdot \\ 0 & 0 & \cdot \\ 0 & 0 & 0 \end{matrix}\right) \left| \begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix} \right.$$

?

$|E| = 0 \iff |A| = 0$

at least one free var.

or a pivot in the last col.