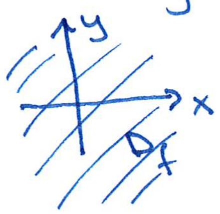
Plan

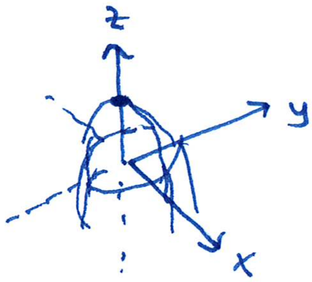
1. Functions in two variables and partial derivatives
2. Unconstrained optimization
3. Constrained optimization and Lagrange multipliers

① Functions in two variables

Ex: $f(x,y) = 2 - x^2 - y^2$

$D_f = \mathbb{R}^2$



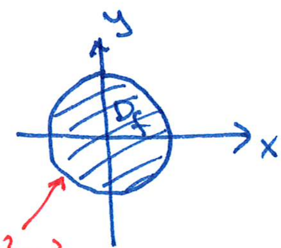


D_f = all pts. (x,y) for which we can compute $f(x,y)$

Graph of f: All pts (x,y,z) such that $z = f(x,y)$ with (x,y) in D_f.

Ex: $f(x,y) = \sqrt{2 - x^2 - y^2}$

$D_f = \{(x,y) : x^2 + y^2 \leq 2\}$

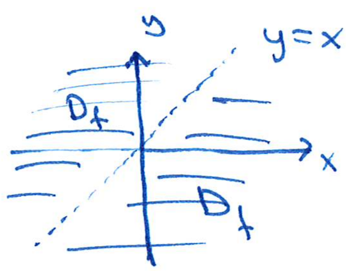


Circle: $(x-x_0)^2 + (y-y_0)^2 = r^2$

$x^2+y^2=2$ circle, center $(0,0)$, $r=\sqrt{2}$

Ex: $f(x,y) = \dfrac{1}{x-y}$

$D_f = \{(x,y) : x - y \neq 0\}$



Partial derivatives:

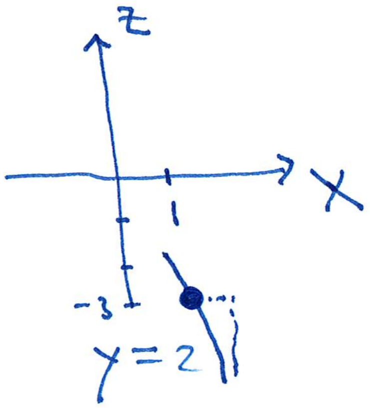
Ex: $f(x,y) = 2 - x^2 - y^2$

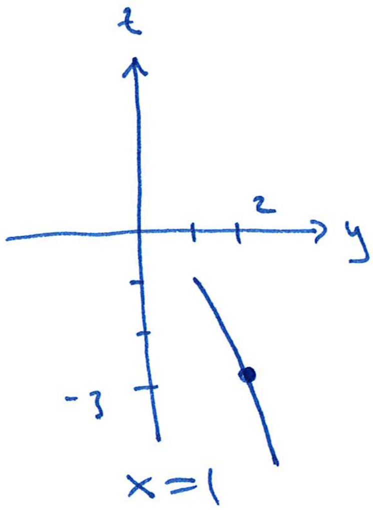
$f'_x(x,y) = 0 - 2x - 0 = \underline{-2x}$ $\quad\quad f'_x = \underline{-2x}$

$f'_y(x,y) = 0 - 0 - 2y = \underline{-2y}$ $\quad\quad f'_y = \underline{-2y}$

Interpretation:

$(x,y) = (1,2)$: $z = f(1,2) = -3$





$f'_x(1,2) = -2 \cdot 1 = -2$ $\quad\quad f'_y(1,2) = -4$

Stationary pts: Pts where $\underline{f'_x = f'_y = 0}$. ← cond. for max/min.

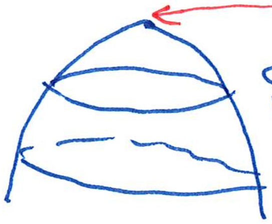
(FOC = first order conditions)

Ex: $f(x,y) = 2 - x^2 - y^2$

$f'_x = -2x = 0 \quad x = 0$

$f'_y = -2y = 0 \quad y = 0$

Stationary pts: $(x,y) = \underline{(0,0)}$



graph of f

(0,0)

In fact, $(0,0)$ is (global) max for f.

$f_{max} = \underline{\underline{2}} = f(0,0)$

Ex: $f(x,y) = x^3 - xy + y^2$

$f'_x = 3x^2 - y \cdot 1 + 0 = 3x^2 - y = 0$

$f'_y = 0 - x \cdot 1 + 2y = -x + 2y = 0 \Rightarrow x = 2y$

$3(2y)^2 - y = 0$

$12y^2 - y = 0$

$y(12y - 1) = 0$

$\underline{y = 0}$ or $12y - 1 = 0$

$\underline{x = 0}$ $\quad y = \underline{1/12}$

$x = 2/12 = \underline{1/6}$

Stationary pts: $(x,y) = \underline{(0,0)}, \ \underline{(1/6, 1/12)}$

$f(0,0) = 0$ $\quad f(1/6, 1/12)$

$= \left(\frac{1}{6}\right)^3 - \frac{1}{6} \cdot \frac{1}{12} + \left(\frac{1}{12}\right)^2$

$= \frac{1}{6^3 \cdot 2^2}(4 - 12 + 6)$

$= -\frac{1}{6^3 \cdot 2}$

(2) Unconstrained optimization

max/min $f(x,y)$

Defn: (x^*,y^*) is a local max for f if $f(x^*,y^*) \geq f(x,y)$ for any (x,y) close to (x^*,y^*)

— " — local min — " — $f(x^*,y^*) \leq f(x,y)$ for any (x,y) close to (x^*,y^*)

Any stationary pt that is not local max/min is called a saddle pt.

Method: – find all stationary pts
– classify them as local max, local min, or saddle pt.
– check if local max are global max
(local min " global min)

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Second derivative test:

For any stationary pt (x^*, y^*) of f, look at

$$H(f)(x^*,y^*) = \begin{pmatrix} f''_{xx}(x^*,y^*) & f''_{xy}(x^*,y^*) \\ f''_{xy}(x^*,y^*) & f''_{yy}(x^*,y^*) \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

Then:
i) $AC-B^2>0$, $A>0$: (x^*,y^*) local min
ii) $AC-B^2>0$, $A<0$: local max
iii) $AC-B^2<0$: saddle pt.

Note: $AC-B^2 = \det H(f)(x^*,y^*)$
If $AC-B^2>0$, then A and C has the same sign.
In this case: $A>0 \Leftrightarrow C>0 \Leftrightarrow \underbrace{A+C}_{\operatorname{tr} H(f)(x^*,y^*)}>0$

Ex: $f(x,y) = x^3 - xy + y^2$

$f'_x = 3x^2 - y$ $\quad f''_{xx} = 6x$ $\quad f''_{xy} = -1$

$f'_y = -x + 2y$ $\quad f''_{yx} = -1$ $\quad f''_{yy} = 2$

Note: If $AC-B^2 = 0$, the second derivative test is 'inconclusive'.

no global max/min

$f(-2,0) = -8 < f(1/6, 1/12) = -\frac{1}{6^3 \cdot 2}$

Ex: $f(x,y) = x^3 - xy + y^2$

Stat. pts: $(0,0)$, $(1/6, 1/12)$ $\quad H(f) = \begin{pmatrix} 6x & -1 \\ -1 & 2 \end{pmatrix}$

$(0,0)$: $H(f)(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$ $\quad A=0,\ B=-1,\ C=2$ $\quad \det = 0 - (-1)^2 = -1 < 0$ saddle pt

$(1/6, 1/12)$: $H(f)(\frac{1}{6}, \frac{1}{12}) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ $\quad A=1,\ B=-1,\ C=2$ $\quad \det = 1\cdot 2 - (-1)^2 = 1 > 0$, $\operatorname{tr} = 1+2>0$ local min

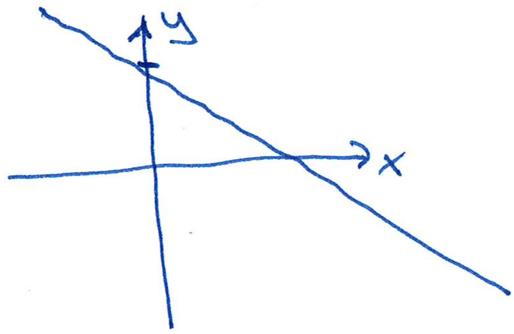
$$\boxed{\begin{array}{l} \max = \text{global max} \\ \min = \text{global min} \end{array}}$$

3 Constrained optimization: Method of Lagrange multipliers

Ex: max/min $\underbrace{f(x,y) = x^2+y^2}_{\text{objective fn.}}$ when $\underbrace{x+3y=10}_{\text{equality constraint}}$

Obj. fn: $f(x,y) = x^2+y^2$

Constr: $\underbrace{x+3y}_{g(x,y)} = \underbrace{10}_{a}$



Method of Lagrange multipliers:

$\mathcal{L}(x,y;\lambda) = f(x,y) - \lambda\,(g(x,y)-a)$

Lagrangian

$x+3y=10$

$3y = -x+10$

$y = -\frac{1}{3}x + \frac{10}{3}$

λ : lambda Lagrange multiplier

$$\begin{aligned} \mathcal{L}'_x = 0 &: \quad \boxed{f'_x - \lambda \cdot g'_x = 0} \\ \mathcal{L}'_y = 0 &: \quad \boxed{f'_y - \lambda \cdot g'_y = 0} \end{aligned}$$

← FOC = first order cond.

FOC+C: Lagrange conditions

$\mathcal{L}'_\lambda = 0 : \quad -1 \cdot (g(x,y)-a) = 0$

$g(x,y) - a = 0$

$\boxed{g(x,y) = a}$ ← C = constraint

Solutions to the Lagrange conditions are the candidates for max / min.

FOC + C
($L'_x = L'_y = 0$)
$g(x,y) = a$

Ex: max/min $f(x,y) = x^2 + y^2$ when $x + 3y = 10$

$$\mathcal{L} = x^2 + y^2 - \lambda(x + 3y - 10)$$

$$\mathcal{L}'_x = 2x - \lambda \cdot 1 = 0$$
$$\mathcal{L}'_y = 2y - \lambda \cdot 3 = 0$$

$$\boxed{\begin{matrix} 2x - \lambda = 0 \\ 2y - 3\lambda = 0 \end{matrix}} \text{ FOC}$$

$$\boxed{x + 3y = 10} \text{ C}$$

Alt I:

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 2 & -3 & 0 \\ 1 & 3 & 0 & 10 \end{array}\right) \cdot 2$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 2 & -3 & 0 \\ 2 & 6 & 0 & 20 \end{array}\right) -1$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 6 & 1 & 20 \end{array}\right) -3$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 10 & 20 \end{array}\right)$$

$2x - \cancel{2\lambda} \lambda = 0 \quad \underline{x = 1}$
$2y - 3\lambda = 0 \quad \underline{y = 3}$
$10\lambda = 20 \quad \underline{\lambda = 2}$

Candidate pts:
$(x,y;\lambda) = \underline{(1,3;2)}$

$f(1,3) = \underline{10}$

Alt 2:

$2x - \lambda = 0 \Rightarrow x = \frac{\lambda}{2}$
$2y - 3\lambda = 0 \quad y = \frac{3\lambda}{2}$

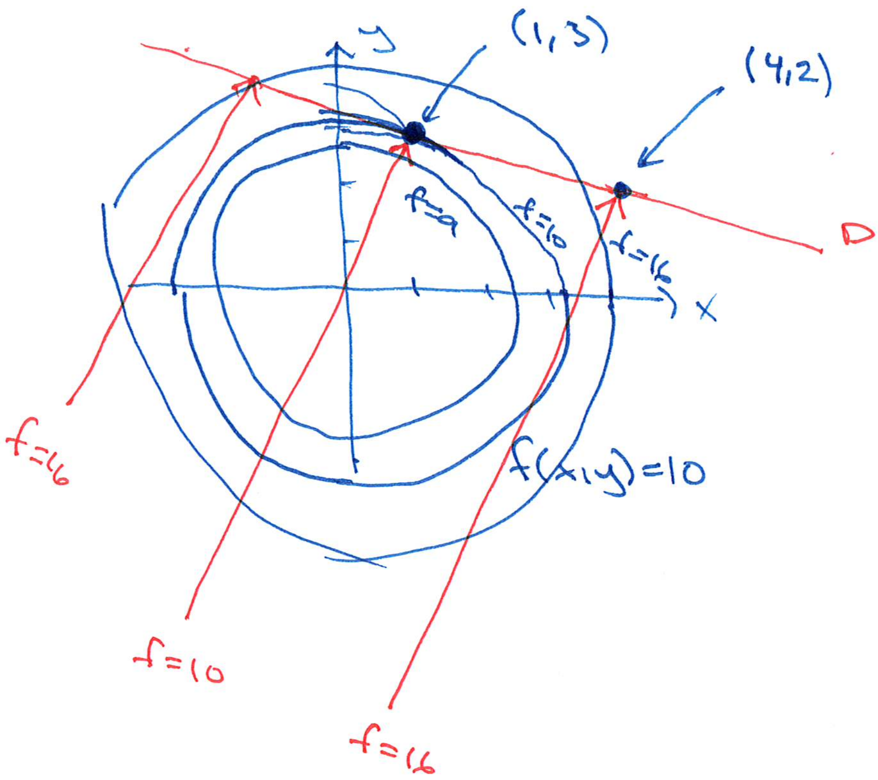
$x + 3y = 10$
$\frac{\lambda}{2} + 3 \cdot \frac{3\lambda}{2} = 10 \quad | \cdot 2$

$\lambda + 9\lambda = 20$
$10\lambda = 20 \quad \underline{\lambda = 2}$

$x = \frac{\lambda}{2} = 1$

$y = \frac{3\lambda}{2} = 3$

$(x,y;\lambda) = \underline{(1,3;2)}$



max/min $f = x^2 + y^2$

when $x + 3y = 10$

$D = \{(x,y) : x + 3y = 10\}$

set of adm. pts.

Level sets for f:

$f(x,y) = c$

$c = 10$: $f(x,y) = 10$

$x^2 + y^2 = 10$

circle, $r = \sqrt{10}$, center (0,0)

$f_{min} = 10$ at $(x,y) = (1,3)$ with $\lambda = 2$

Interpretation of $\lambda = 2$:

the marginal change in the minimum value ~~when we~~ per unit change the constant a in the constraint $g(x,y) = a$.

Ex: max/min $f(x,y) = x+y$ when $\underline{x^3 - 3xy + y^3 = 0}$

Lagrange:

$L = x + y - \lambda(x^3 - 3xy + y^3)$

$$\boxed{\begin{aligned} L'_x &= 1 - \lambda \cdot (3x^2 - 3y) = 0 \\ L'_y &= 1 - \lambda(-3x + 3y^2) = 0 \\ & x^3 - 3xy + y^3 = 0 \end{aligned}}$$

Cand. pt:

$(x, y; \lambda) = (3/2, 3/2; 4/9)$

$f = 3$

(1) $1 = 3\lambda(x^2 - y)$ $\qquad 3\lambda = \dfrac{1}{x^2 - y}$

(2) $1 = 3\lambda(-x + y^2)$ $\qquad 3\lambda = \dfrac{1}{-x + y^2}$

(3) $x^3 - 3xy + y^3 = 0$

$\Rightarrow \dfrac{1}{x^2 - y} = \dfrac{1}{y^2 - x}$

$y^2 - x = x^2 - y$

$y^2 - x^2 - x + y = 0$

$(y - x)(y + x) + (y - x) = 0$

$(y - x)(y + x + 1) = 0$

$\underline{y = x}$ or $\underline{x + y = -1} \to y = -1 - x$

$y = x$:

$x^3 - 3x^2 + x^3 = 0$

$2x^3 - 3x^2 = 0$

$x^2(2x - 3) = 0$

$\cancel{x = 0}$, $\cancel{y = 0}$, $\cancel{\lambda =}$ or $\boxed{x = 3/2,\ y = 3/2,\ \lambda = 4/9}$

$\lambda = \dfrac{1}{3} \cdot \dfrac{1}{x^2 - y}$

$= \dfrac{1}{3} \cdot \dfrac{1}{\frac{9}{4} - \frac{3}{2}} \cdot \dfrac{4}{4}$

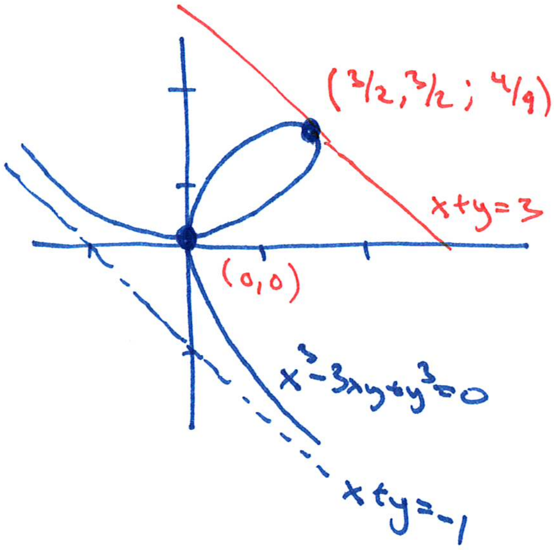
$= \dfrac{4}{3 \cdot 3} = \dfrac{4}{9}$

$x + y = -1$:

$x^3 - 3x(-1-x) + (-1-x)^3 = 0$

$\cancel{x^3} + \cancel{3x} + \cancel{3x^2} + (-1) - \cancel{3x} - \cancel{3x^2} - \cancel{x^3} = 0$

$-1 = 0$



max/min $f(x,y) = x + y$ when $x^3 - 3xy + y^3 = 0$

$V_f = (-1, 3]$

$f_{max} = 3$

f_{min} : no min