EXTRA LECTURE 1: GRA 6035

MATHEMATICS

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① INTRODUCTION

② STRATEGY

③ BASICS OF OPTIMIZATION
(We did ⓐ - ⓒ below; ⓓ - ⓔ will be done Jan 26)

~70% of exam questions

①-② FOCUS ON BASIC PROBLEMS

ONE TYPE OF PROBLEM FOR EACH LECTURE.

ADVANCED PROBLEMS IN THE LAST LECTURES

IF THERE IS TIME.

remaining ~30% of exam questions

③ BASICS OF OPTIMALIZATION

ⓐ derivation, computation of the Hessian

ⓑ find stationary points

ⓒ classify types of stationary points

25/01 Extra Lecture 1

ⓓ determine if a function is convex / concave

ⓔ find pts that satisfy first order conditions + constraints in constrained optim. problems.

26/01 Extra Lecture 2

(a) Derivation, Hessian matrix.

Find f'_x, f'_y, f'_z and the Hessian matrix in these cases:

i) $f = xy + xz - yz$

ii) $f = x^2 + y^2 + z^2 + z^3 + 2yz - 2x + 12y$

iii) $f = x^2 + 4xy + 4y^2 + e^y - y$

iv) $f = x^2 + y^2 + y^4 + yz - 1$

v) $f = \ln(x+1) + \ln(y+1) - \ln(z-1)$

vi) $f = z \cdot \sqrt{x^2+y^2}$

vii) $f = e^{xyz}$

Solution:

i) $f'_x = y+z$

$f'_y = x-z$

$f'_z = x-y$

$$H(f) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

ii) $f'_x = 2x - 2$

$f'_y = 2y + 2z + 12$

$f'_z = 2z + 3z^2 + 2y$

$$H(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2+6z \end{pmatrix}$$

iii) $f'_x = 2x + 4y$

$f'_y = 4x + 8y + e^y - 1$

$f'_z = 0$

$$H(f) = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 8+e^y & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

iv) $f'_x = 2x$

$f'_y = 2y + 4y^3 + z$

$f'_z = y$

$$H(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+12y^2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

v) $f'_x = \dfrac{1}{x+1} = (x+1)^{-1}$

$f'_y = \dfrac{1}{y+1} = (y+1)^{-1}$

$f'_z = -\dfrac{1}{z-1} = -(z-1)^{-1}$

$$H(f) = \begin{vmatrix} -\dfrac{1}{(x+1)^2} & 0 & 0 \\ 0 & -\dfrac{1}{(y+1)^2} & 0 \\ 0 & 0 & +\dfrac{1}{(z-1)^2} \end{vmatrix}$$

vi) $f'_x = z \cdot \dfrac{1 \cdot 2x}{2\sqrt{x^2+y^2}} = \dfrac{xz}{\sqrt{x^2+y^2}}$

$f'_y = z \cdot \dfrac{1 \cdot 2y}{2\sqrt{x^2+y^2}} = \dfrac{yz}{\sqrt{x^2+y^2}}$

$f'_z = \sqrt{x^2+y^2} = \sqrt{x^2+y^2}$

$$H(f) = \begin{pmatrix} \dfrac{y^2 z}{(x^2+y^2)^{3/2}} & \dfrac{-xyz}{(x^2+y^2)^{3/2}} & \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{-xyz}{(x^2+y^2)^{3/2}} & \dfrac{x^2 z}{(x^2+y^2)^{3/2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} & 0 \end{pmatrix}$$

not basic problem

vii) $f'_x = e^{xyz} \cdot yz$

$f'_y = e^{xyz} \cdot xz$

$f'_z = e^{xyz} \cdot xy$

$$H(f) = \begin{pmatrix} y^2z^2 & z(xyz+1) & y(xyz+1) \\ z(xyz+1) & x^2z^2 & x(xyz+1) \\ y(xyz+1) & x(xyz+1) & x^2y^2 \end{pmatrix} \cdot e^{xyz}$$

not basic problem

v) <u>In detail</u>:

$$f(x,y,z) = \ln(x+1) + \ln(y+1) - \ln(z-1)$$

$$\left(\ln(x+1)\right)'_x = \left(\ln(u)\right)'_x = \frac{1}{u} \cdot u'_x = \frac{1}{x+1} \cdot 1$$

$u = x+1$

vi) <u>In detail</u>: $f = z \cdot \sqrt{x^2+y^2} = z \cdot \sqrt{u}$, $\underline{u = x^2+y^2}$

$$f'_x = z\,(\underset{\substack{\| \\ u^{1/2}}}{\sqrt{u}})'_x = z \cdot \frac{1}{2\sqrt{u}} \cdot u'_x = z \cdot \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}x$$

$$= \frac{xz}{\sqrt{x^2+y^2}}$$

$$f'_y = \frac{yz}{\sqrt{x^2+y^2}}$$ (in the same way as above)

$$f'_z = \sqrt{x^2+y^2} \cdot 1$$

The Hessian in vi) : $f = z\cdot\sqrt{x^2+y^2}$

$$f'_x = \frac{xz}{\sqrt{x^2+y^2}} = \frac{u}{v} \qquad f'_y = \frac{yz}{\sqrt{x^2+y^2}} \qquad f'_z = \sqrt{x^2+y^2}$$

$$f''_{xx} = \frac{u'_x v - u v'_x}{v^2} = \frac{\left(z\cdot\sqrt{x^2+y^2} - xz\cdot\frac{1\cdot\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}\right)\cdot\sqrt{x^2+y^2}}{(x^2+y^2)\cdot\sqrt{x^2+y^2}}$$

$$= \frac{z\cdot(\cancel{x^2}+y^2) - \cancel{x^2 z}}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{y^2 z}{(x^2+y^2)^{3/2}}$$

$$f''_{zx} = \frac{1\cdot 2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$f''_{zy} = \frac{y}{\sqrt{x^2+y^2}}$$

$$f''_{zz} = 0$$

can be computed in a similar way

$$H(f) = \begin{pmatrix} \frac{y^2 z}{(x^2+y^2)^{3/2}} & * & \frac{x}{\sqrt{x^2+y^2}} \\ * & * & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \end{pmatrix}$$

vii) in detail: $f = e^{xyz} = e^{u}$, $u = xyz$

$$f'_x = (e^u)'_x = e^u \cdot u'_x = e^{xyz} \cdot yz$$

$$f'_y = (e^u)'_y = e^u \cdot u'_y = e^{xyz} \cdot xz$$

$$f'_z = (e^u)'_z = e^u \cdot u'_z = e^{xyz} \cdot xy$$

$$f''_{xx} = yz \cdot (e^{xyz})'_x = yz \cdot e^{xyz} \cdot yz = \underline{y^2 z^2 e^{xyz}}$$

$$f''_{xy} = (\underset{u}{e^{xyz}} \cdot \underset{v}{yz})'_y = \underset{u'}{(e^{xyz})'_y} \cdot \underset{v}{yz} + \underset{u}{e^{xyz}} \cdot \underset{v'}{(yz)'_y}$$

$$= e^{xyz} \cdot xz \cdot yz + e^{xyz} \cdot z = e^{xyz}(xyz^2 + z)$$

$$= \underline{e^{xyz} \cdot z \cdot (xyz+1)}$$

rest of the second order derivatives can be computed in a similar way

$$H(f) = \begin{pmatrix} y^2z^2 & z(xyz+1) & y(xyz+1) \\ z(xyz+1) & x^2z^2 & x(xyz+1) \\ y(xyz+1) & x(xyz+1) & x^2y^2 \end{pmatrix} e^{xyz}$$

(b) Stationary pts

Solve: $f'_x = 0 \qquad f'_y = 0 \qquad f'_z = 0$

Tip: Solve easy equations first!

Exercise: Find st. pts in i) - vii) above.

i)
$$f'_x = y + z = 0$$
$$\left.\begin{array}{ll} f'_y = x - z = 0 & \Rightarrow x = z \\ f'_z = x - y = 0 & \Rightarrow x = y \end{array}\right\} \Rightarrow x = y = z$$

$$\Rightarrow y + z = y + y = 0$$
$$2y = 0$$
$$y = 0$$
$$\Downarrow$$
$$\underline{\underline{x = y = z = 0}}$$

ii)
$$f'_x = 2x - 2 = 0 \qquad \Rightarrow x = 1$$
$$f'_y = 2y + 2z + 12 = 0 \qquad \Rightarrow 2y + 2z = -12$$
$$y + z = -6 \qquad y = -6 - z$$
$$f'_z = 2z + 3z^2 + 2y = 0$$
$$\Rightarrow 2z + 3z^2 + 2(-6 - z) = 0$$
$$3z^2 - 12 = 0$$
$$z^2 = 4 \Rightarrow z = \pm 2$$
$$y = -8, -4$$
$$\Downarrow$$
$$(x, y, z) = \cancel{(1, 2, -8) \text{ or } (1, -2, -4)} \quad (1, -8, 2) \text{ or } \underline{\underline{(1, -4, -2)}}$$

iii) $f'_x = 2x+4y=0 \quad \Rightarrow \quad x=-2y \quad \Rightarrow x=0$

$f'_y = 4x+8y+e^y-1=0 \quad \Rightarrow \quad 4(-2y)+8y+e^y-1=0 \Rightarrow e^y=1 \Rightarrow y=0$

$(y=\ln 1)$

$f'_z = 0 \quad \Rightarrow z$ free variable

$\Downarrow$

$(x,y,z) = \underline{\underline{(0,0,z)}}$ (z free variable)

iv) $f'_x = 2x = 0 \quad \Rightarrow x=0$

$f'_y = 2y+4y^3+z=0 \quad \Rightarrow z=0$

$f'_z = y = 0 \quad \Rightarrow y=0$

$\Rightarrow (x,y,z) = \underline{\underline{(0,0,0)}}$

v) $f'_x = \frac{1}{x+1} = 0$ impossible $\Rightarrow$ <u>no solutions</u>

$f'_y = \frac{1}{y+1} = 0$

$\Downarrow$

$f'_z = -\frac{1}{z-1} = 0$

<u>no stationary pts</u>

vi) $f'_x = \frac{xz}{\sqrt{x^2+y^2}} = 0$

$f'_y = \frac{yz}{\sqrt{x^2+y^2}} = 0$

$f'_z = \sqrt{x^2+y^2} = 0 \quad \Rightarrow x=0, y=0 \Rightarrow f'_x, f'_y$ not defined

$\Rightarrow$ <u>no stationary pts.</u>

vii) $f'_x = e^{xyz} \cdot yz = 0 \implies yz = 0$

$f'_y = e^{xyz} \cdot xz = 0 \implies xz = 0$

$f'_z = e^{xyz} \cdot xy = 0 \implies xy = 0$

$yz = 0 \implies \boxed{y = 0 \quad \text{or} \quad z = 0}$

$\underline{y=0}$: $\underset{\text{ok}}{yz=0}, \underset{\substack{| \\ x=0 \text{ or } z=0}}{xz=0}, \underset{\text{ok}}{xy=0} \Bigg\} \implies \begin{matrix} y=0, x=0 \\ \text{or} \\ y=0, z=0 \end{matrix}$

$\underline{y \neq 0}$: $\underset{\substack{\Downarrow \\ z=0}}{yz=0}, \underset{\text{ok}}{xz=0}, \underset{\substack{\Downarrow \\ x=0}}{xy=0} \implies x=0, z=0$

$\underline{\text{Stat. pts}}$: $\begin{cases} (0,0,z) & z \text{ free} \\ (0,y,0) & y \text{ free} \\ (x,0,0) & x \text{ free} \end{cases}$

(c) Classification of stationary points

Look at the Hessian matrix at the stationary point;

$$f''(x_0, y_0, z_0)$$

where (x_0, y_0, z_0) is the stationary point.

Repeat for each stationary point

positive definite	$\Rightarrow$ local min.
negative definite	$\Rightarrow$ local max.
indefinite	$\Rightarrow$ saddle point

Classify all stationary pts in i) – vii).

i) Stationary pts: $(0,0,0)$

Hessian at $(0,0,0)$:

$$f''(0,0,0) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$D_1 = 0$
$D_2 = -1 \Rightarrow$ indefinite

$\Downarrow$

$(0,0,0)$ is saddle pt.

ii) Stationary pts: $(1,-8,2)$, $(1,-4,-2)$

$$f''(1,8,+2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 14 \end{pmatrix}$$

$D_1 = 2$
$D_2 = 4$
$D_3 = 48$

positive defn.

$\Downarrow$

$(1,-8,2)$ local min

$$f''(1,-4,-2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & -10 \end{pmatrix}$$

$D_1 = 2$
$D_2 = 4$
$D_3 = -48$

indefinite

$\Downarrow$

$(1,-4,-2)$ saddle point

We use (leading) principal minors:

$D_1 > 0,\ D_2 > 0,\ D_3 > 0 \iff$ positive definite

$D_1 < 0,\ D_2 > 0,\ D_3 < 0 \iff$ negative definite

Doesn't fit pattern
$\boxed{\begin{matrix} D_1 \geq 0,\ D_2 \geq 0,\ D_3 \geq 0 \\ \text{or} \\ D_1 \leq 0,\ D_2 \geq 0,\ D_3 \leq 0 \end{matrix}}$ $\iff$ indefinite

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Problems (Contined)

iii) Stat. pts: $(0,0,z)$ (z free)

Hessian:

$$f''(0,0,z) = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$D_1 = 2$ $\quad \Delta_1 = 2, 9, 0$

$D_2 = 18 - 16 = 2$ $\quad \Delta_2 = 2, 0, 0$

$D_3 = 0$

positive semidefinite

$\Downarrow$

second derivative test is inconclusive

We must use another method:

* $f(x,y) = x^2 + 4xy + 4y^2 + e^y - y$ is positive definite as a function of two vars

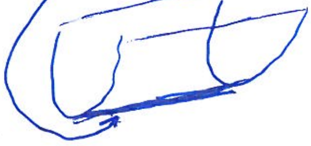
  stat pts: $(0,0)$

  Hessian: $\begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix}$ $\quad D_1 = 2$, $D_2 = 2$

  picture 

not basic problem

Each $(0,0,z)$ is a local min



* When we extend to three vars, but z is not part of the ... get this picture :

iv) $(x,y,z) = (0,0,0)$ st. pts.

Hessian:

$$f''(0,0,0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$D_1 = 2$
$D_2 = 0$
$D_3 = -2$

indefinite
$\Downarrow$
$(0,0,0)$ is saddle pt

v), vi) No stationary pts

vii) Stat. pts: $(x,0,0)$ x free
$(0,y,0)$ y "
$(0,0,z)$ z "

$$f''(x,0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & x & 0 \end{pmatrix}$$

$D_1 = 0$ $\quad \Delta_1 = 0,0,0$
$D_2 = 0$ $\quad \Delta_2 = 0,0,-x^2$
$D_3 = 0$ $\quad \Delta_3 = 0$

$\Rightarrow$ $(x,0,0)$ saddle pt if $x \neq 0$
test inconclusive if $x = 0$

$$f''(0,y,0) = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ y & 0 & 0 \end{pmatrix}$$

$D_1 = 0$ $\quad \Delta_1 = 0,0,0$
$D_2 = 0$ $\quad \Delta_2 = 0,-y^2,0$
$D_3 = 0$ $\quad \Delta_3 = 0$

$\Rightarrow$ $(0,y,0)$ saddle pt if $y \neq 0$
test inconclusive if $y = 0$.

$$f''(0,0,z) = \begin{pmatrix} 0 & z & 0 \\ z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$D_1 = 0$ $\quad \Delta_1 = 0,0,0$
$D_2 = -z^2$ $\quad \Delta_2 = -z^2,0,0$
$D_3 = 0$ $\quad \Delta_3 = 0$

$\Rightarrow$ $(0,0,z)$ saddle pt if $z \neq 0$
test inconclusive if $z = 0$

$(x,y,z)=(0,0,0)$: Also saddle point, we must use a different method to show this:

not basic problem $\rightarrow$

$f(0,0,0) = e^0 = 1$

$f(a,a,a) = e^{a^3} > e^0 = 1$ if $a>0$

$e^{a^3} < e^0 = 1$ if $a<0$

$\Downarrow$

$(0,0,0)$ saddle point

Tomorrow 26/01 : EXTRA LECTURE 2

- continue with
  - d) convex/concave functions
  - e) find pts that satisfy F.O.C. + Constraints.
- Start with matrix problems